2018 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: **10:00 to 12:30**

This English translation is supplemental and provided for convenience of applicants. The Japanese version is the official one.

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may use the back of the sheet if necessary.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. Do not separate the draft papers from this problem booklet.
7. Any answer sheet including marks, symbols and/or words unrelated to your answer will be invalid.
8. Do not take either the answer sheets or the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

(draft paper)

Problem 1

Consider to solve the following simultaneous linear equation:

$$\boldsymbol{Ax} = \boldsymbol{b}$$

where $\boldsymbol{A} \in \mathcal{R}^{m \times n}$, $\boldsymbol{b} \in \mathcal{R}^m$ are a constant matrix and a vector, and $\boldsymbol{x} \in \mathcal{R}^n$ is an unknown vector. Answer the following questions.

(1) An $m \times (n+1)$ matrix $\overline{\boldsymbol{A}} = (\boldsymbol{A} \mid \boldsymbol{b})$ is made by adding a column vector after the last column of matrix $\boldsymbol{A}$. In the case of $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\overline{\boldsymbol{A}} = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 4 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ is obtained. Let the i-th column vector of the matrix $\overline{\boldsymbol{A}}$ be $\boldsymbol{a}_i$ ($i = 1, 2, 3, 4$).

(i) Find the maximum number of linearly independent vectors among $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$.

(ii) Show that $\boldsymbol{a}_4$ can be represented as a linear sum of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, by obtaining scalars x_1 and x_2 that satisfy $\boldsymbol{a}_4 = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \boldsymbol{a}_3$.

(iii) Find the maximum number of linearly independent vectors among $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$.

(2) Show that the solution of the simultaneous linear equation exists when $\operatorname{rank}\overline{\boldsymbol{A}} = \operatorname{rank}\boldsymbol{A}$, for arbitrary m, n, $\boldsymbol{A}$ and $\boldsymbol{b}$.

(3) There is no solution when $\operatorname{rank}\overline{\boldsymbol{A}} > \operatorname{rank}\boldsymbol{A}$. When $m > n$, $\operatorname{rank}\boldsymbol{A} = n$ and $\operatorname{rank}\overline{\boldsymbol{A}} > \operatorname{rank}\boldsymbol{A}$, obtain $\boldsymbol{x}$ that minimizes the squared norm of the difference between the left hand side and the right hand side of the simultaneous linear equation, namely $\| \boldsymbol{b} - \boldsymbol{Ax} \|^2$.

(4) When $m < n$ and $\operatorname{rank}\boldsymbol{A} = m$, there exist multiple solutions for the simultaneous linear equation for arbitrary $\boldsymbol{b}$. Obtain $\boldsymbol{x}$ that minimizes $\| \boldsymbol{x} \|^2$ among them, by adopting the method of Lagrange multipliers and using the simultaneous linear equation as the constraint condition.

(5) Show that there exists a unique $\boldsymbol{P} \in \mathcal{R}^{n \times m}$ that satisfies the following four equations for arbitrary m, n and $\boldsymbol{A}$.

$$\begin{aligned} \boldsymbol{APA} &= \boldsymbol{A} \\ \boldsymbol{PAP} &= \boldsymbol{P} \\ (\boldsymbol{AP})^T &= \boldsymbol{AP} \\ (\boldsymbol{PA})^T &= \boldsymbol{PA} \end{aligned}$$

(6) Show that both $\boldsymbol{x}$ obtained in (3) and $\boldsymbol{x}$ obtained in (4) are represented in the form of $\boldsymbol{x} = \boldsymbol{Pb}$.

Problem 2

Let f_1 be a positive constant function on $[0,1]$ with $f_1(x) = c$, and let p and q be positive real numbers with $1/p + 1/q = 1$. Moreover, let $\{f_n\}$ be the sequence of functions on $[0,1]$ defined by

$$f_{n+1}(x) = p \int_0^x (f_n(t))^{1/q}\,\mathrm{d}t \qquad (n = 1, 2, \ldots).$$

Answer the following questions.

(1) Let $\{a_n\}$ and $\{c_n\}$ be the sequences of real numbers defined by $a_1 = 0$, $c_1 = c$ and

$$a_{n+1} = q^{-1} a_n + 1 \qquad (n = 1, 2, \ldots),$$

$$c_{n+1} = \frac{p\,(c_n)^{1/q}}{a_{n+1}} \qquad (n = 1, 2, \ldots).$$

Show that $f_n(x) = c_n x^{a_n}$.

(2) Let g_n be the function on $[0,1]$ defined by $g_n(x) = x^{a_n} - x^p$ for $n \geq 2$. Noting that $a_n \geq 1$ holds true for $n \geq 2$, show that g_n attains its maximum at a point $x = x_n$, and find the value of x_n.

(3) Show that $\lim_{n\to\infty} g_n(x) = 0$ for any $x \in [0,1]$.

(4) Let d_n be defined by $d_n = (c_n)^{q^n}$. Show that d_{n+1}/d_n converges to a finite positive value as $n \to \infty$. You may use the fact that $\lim_{t\to\infty} (1 - 1/t)^t = 1/\mathrm{e}$.

(5) Find the value of $\lim_{n\to\infty} c_n$.

(6) Show that $\lim_{n\to\infty} f_n(x) = x^p$ for any $x \in [0,1]$.

Problem 3

Let z_n and w_n $(n = 0, 1, 2, \ldots)$ be complex numbers. Consider a bag that contains two red cards and one white card. First, take one card from the bag and return it to the bag. z_{k+1} $(k = 0, 1, 2, \ldots)$ is generated in the following manner based on the color of the card taken.

$$z_{k+1} = \begin{cases} iz_k & \text{if a red card was taken,} \\ -iz_k & \text{if a white card was taken.} \end{cases}$$

Next, take one card from the bag again and return it to the bag. w_{k+1} is generated in the following manner based on the color of the card taken.

$$w_{k+1} = \begin{cases} -iw_k & \text{if a red card was taken,} \\ iw_k & \text{if a white card was taken.} \end{cases}$$

Here, each card is independently taken with equal probability. The initial state is $z_0 = 1$ and $w_0 = 1$. Thus, z_n, w_n are the values after repeating the series of the above two operations n times starting from the state of $z_0 = 1$ and $w_0 = 1$. Here, i is the imaginary unit.

Answer the following questions.

(1) Show that $\mathrm{Re}(z_n) = 0$ if n is odd, and that $\mathrm{Im}(z_n) = 0$ if n is even. Here, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ represent the real part and the imaginary part of z respectively.

(2) Let P_n be the probability of $z_n = 1$, and Q_n be the probability of $z_n = i$. Find recurrence equations for P_n and Q_n.

(3) Find the probabilities of $z_n = 1$, $z_n = i$, $z_n = -1$, and $z_n = -i$ respectively.

(4) Show that the expected value of z_n is $(i/3)^n$.

(5) Find the probability of $z_n = w_n$.

(6) Find the expected value of $z_n + w_n$.

(7) Find the expected value of $z_n w_n$.

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