数理情報学専攻 修士課程入学試験問題

Department of Mathematical Informatics

Graduate School Entrance Examination Problem Booklet

専門科目 数理情報学

Specialized Subject: Mathematical Informatics

2025年8月20日（水）　10:00 – 13:00

August 20, 2025 (Wednesday)　10:00 – 13:00

5問出題，3問解答 / Answer 3 out of the 5 problems

注　意　事　項 / Instructions

(1) 試験開始の合図まで，この問題冊子を開かないこと．
Do not open this booklet until the starting signal is given.

(2) 本冊子に落丁，乱丁，印刷不鮮明の箇所などがあった場合には申し出ること．
Notify the proctor if there are missing or incorrect pages in your booklet.

(3) 本冊子には第1問から第5問まであり，日本文は4頁から13頁，英文は14頁から23頁である．5問のうち3問を日本語ないし英語で解答すること．
Five problems appear on pages 4–13 in Japanese and pages 14–23 in English in this booklet. Answer 3 problems in Japanese or English.

(4) 答案用紙3枚が渡される．1問ごとに必ず1枚の答案用紙を使用すること．止むを得ぬときは答案用紙の裏面を使用してもよい．
Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(5) 各答案用紙の指定された箇所に，受験番号およびその用紙で解答する問題番号を忘れずに記入すること．氏名は書いてはならない．
Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(6) 草稿用紙は本冊子から切り離さないこと．
Do not separate a draft sheet from the booklet.

(7) 解答に関係のない記号，符号などを記入した答案は無効とする．
Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(8) 答案用紙および問題冊子は持ち帰らないこと．
Leave the answer sheets and this booklet in the examination room.

受験番号 Examinee number	No.

上欄に受験番号を記入すること．

Fill in your examinee number.

選択した問題番号 Problem numbers			

上欄に選択した3つの問題番号を記入すること．

Fill in the three selected problem numbers.

Problem 1

Let n be a positive integer. Let $\mathbb{S}^n$ denote the set of all $n \times n$ real symmetric matrices, and let $\mathbb{S}^n_+$ denote the set of all $n \times n$ real symmetric positive semidefinite matrices. For matrices $A, B \in \mathbb{S}^n$, let $\mathrm{tr}(A)$ denote the sum of the diagonal entries of A, and let $\langle A, B \rangle = \mathrm{tr}(AB)$ and $\|A\| = \sqrt{\langle A, A \rangle}$. Answer the following questions.

(1) Show that $\alpha X + (1 - \alpha) Y \in \mathbb{S}^n_+$ for any $X, Y \in \mathbb{S}^n_+$ and any $\alpha \in [0, 1]$.

(2) Show that there exists a unique $X \in \mathbb{S}^n_+$ that minimizes $\|X - A\|^2$ for a given $A \in \mathbb{S}^n$. Let $\phi(A) \in \mathbb{S}^n_+$ be the minimizer. Write $\phi(A)$ in terms of the eigenvalues and eigenvectors of A.

(3) Show that

$$\langle A - \phi(A), X - \phi(A) \rangle \leq 0$$

for any $A \in \mathbb{S}^n$ and any $X \in \mathbb{S}^n_+$.

(4) Show that the following (i) and (ii) are equivalent for any $X, Y \in \mathbb{S}^n_+$.

(i) $\langle X, Y \rangle = 0$ holds.

(ii) $X = \phi(X - Y)$ holds.

Problem 2

Let $\mathbb{R}$ be the set of all real numbers. For a real number x, denote $\exp(x) = \mathrm{e}^x$. The expectation of a real-valued random variable Z is denoted by $\mathbb{E}[Z]$. The moment generating function M_Z of Z is defined by $M_Z(\lambda) = \mathbb{E}[\exp(\lambda Z)]$ for $\lambda \in \mathbb{R}$. Answer the following questions.

(1) For a random variable X that follows the standard normal distribution, define $Y = X^2$. Show that

$$M_Y(\lambda) = \frac{1}{\sqrt{1-2\lambda}}$$

holds for any $\lambda < 1/2$.

(2) Show that

$$\Pr(Z \geq \varepsilon) \leq \frac{M_Z(\lambda)}{\exp(\lambda\varepsilon)}$$

holds for any real-valued random variable Z, any positive real number ε, and any $\lambda > 0$ such that $M_Z(\lambda)$ is finite.

In the following, let m and n be positive integers, and let A be an $m \times n$ matrix whose entries are independently following the standard normal distribution. For a positive integer ℓ, the Euclidean norm of $b \in \mathbb{R}^\ell$ is denoted by $\|b\|$.

(3) Show that

$$\Pr(\|Ab\|^2 - m \geq \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{8m}\right)$$

holds for any $b \in \mathbb{R}^n$ satisfying $\|b\| = 1$ and for any $\varepsilon \in (0, m]$. Here, you may use the fact that $\log_\mathrm{e}(1+x) \geq x - x^2$ holds for any $x \geq -1/2$.

(4) Let k be an integer satisfying $k \geq 2$, and let $\{b_1, b_2, \ldots, b_k\} \subseteq \mathbb{R}^n$. For each $i = 1, \ldots, k$, let $b'_i = \frac{1}{\sqrt{m}} A b_i$. Show that, if

$$m \geq \frac{8}{\eta^2} \log_\mathrm{e}\left(\frac{k(k-1)}{2\delta}\right)$$

holds for constants $\eta, \delta \in (0, 1]$, then the probability of holding the inequalities

$$\|b'_i - b'_j\|^2 \leq (1+\eta)\|b_i - b_j\|^2 \quad \text{for all } i, j \in \{1, \ldots, k\}$$

is greater than or equal to $1 - \delta$.

Problem 3

Consider the following differential equations with respect to the two real-valued functions $x(t)$ and $y(t)$:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2xy, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 2xy - y. \end{cases} \tag{$*$}$$

Here, we assume that an initial condition $(x(0), y(0))$ for the equations $(*)$ is given, and the solution is unique and sufficiently smooth on the interval where the solution exists. Answer the following questions.

(1) Find all $(x(0), y(0))$ satisfying

$$\text{“}(x(t), y(t)) = (x(0), y(0)) \text{ holds for all } t \geq 0\text{”}.$$

(2) Suppose the initial condition satisfies $x(0) > 0, y(0) > 0$, and $x(0) + y(0) = 1$. Show that under this assumption the solution of the equations $(*)$ satisfies

$$x(t) \geq 0,\ y(t) \geq 0, \text{ and } x(t) + y(t) \leq 1$$

for all $t \geq 0$. You can assume that the solution exists in $t \geq 0$.

(3) We assume the same assumption for the initial condition as in (2). Discuss whether there exists a solution to the equations $(*)$ which satisfies

$$\frac{1}{2} \leq \lim_{t\to\infty} x(t) < 1$$

or not.

(4) Suppose the initial condition satisfies $(x(0), y(0)) = (1, -1)$. Show that there exists a positive T such that $\lim\limits_{\substack{t\to T \\ t<T}} x(t) = \infty$ (i.e., $x(t)$ diverges to infinity as t approaches T from the left). You may use the following.

$$\text{If } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0, \text{ then } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \text{ holds.}$$

Problem 4

Let N be a positive integer satisfying $N \geq 2$. Consider a population of N individuals, where each individual in the population has a type of either A or B. The fitness of type A (resp. type B) is represented by a positive real number f_{A} (resp. f_{B}), and we define $r = f_{\mathrm{A}}/f_{\mathrm{B}}$. The number $X_0 \in \{0, \ldots, N\}$ of individuals of type A at time 0 in the population is given. For $t = 0, 1, \ldots$, the number X_{t+1} of individuals of type A at time $t+1$ is determined by using the number X_t of individuals of type A at time t as follows:

- One individual is selected from N individuals with equal probability $1/N$. Let u_t be a random variable following the uniform distribution defined on the open interval $(0, 1)$.
  - If the selected individual is of type A and
  $$u_t > \frac{f_{\mathrm{A}} X_t}{f_{\mathrm{A}} X_t + f_{\mathrm{B}}(N - X_t)}$$
  holds, then the type of the selected individual changes to B.
  - If the selected individual is of type B and
  $$u_t \leq \frac{f_{\mathrm{A}} X_t}{f_{\mathrm{A}} X_t + f_{\mathrm{B}}(N - X_t)}$$
  holds, then the type of the selected individual changes to A.
  - The type of the selected individual does not change, otherwise.
- The types of the unselected $N - 1$ individuals do not change.
- Let X_{t+1} be the number of individuals of type A after this type transition.

Here, the selection of individuals at time $t = 0, 1, \ldots$, and $u_0, u_1, \ldots$ are independent.

For $i = 1, \ldots, N-1$, we denote $p_i = \Pr(X_1 = i + 1)$ and $q_i = \Pr(X_1 = i - 1)$ in the case when $X_0 = i$. Answer the following questions.

(1) Show that $p_i/q_i = r$ holds for $i = 1, \ldots, N - 1$.

(2) For $i = 0, \ldots, N$, we define
$$\phi_i = \Pr\left(\lim_{t\to\infty} X_t \text{ exists and is equal to } N\right)$$
in the case when $X_0 = i$. Then for $i = 1, \ldots, N - 1$, provide a relation of ϕ_{i+1}, ϕ_i, and ϕ_{i-1} in terms of p_i and q_i.

(3) Write ϕ_i in terms of i, N, r for $i = 0, \ldots, N$.

(4) For $i = 0, \ldots, N$, let T_i be the expectation of the first time that X_t becomes either 0 or N starting from $X_0 = i$. The probability that no such time t exists is equal to 0. Then, for $i = 1, \ldots, N-1$, provide a relation of T_{i+1}, T_i, and T_{i-1} in terms of p_i and q_i. You may use that $T_i < \infty$ for $i = 0, \ldots, N$.

(5) Suppose $r = 1$. For $i = 1, \ldots, N-1$, write T_i in terms of i, N, L_N, L_{i+1}, and L_{N-i+1}. Here, we define $L_1 = 0$ and $L_j = \sum_{k=1}^{j-1} 1/k$ for $j \geq 2$. You may use that $\sum_{j=1}^{n} L_j = nL_{n+1} - n$ for any positive integer n.

Problem 5

Let $G = (V, E)$ be a simple undirected graph with a set $V = \{v_1, v_2, \ldots, v_n\}$ of n vertices and a set E of m edges, and suppose that G is given by an adjacency list representation. An orientation of G is a directed graph $\vec{G} = (V, \vec{E})$ obtained from G by giving a direction to each edge. In particular, if $\vec{G} = (V, \vec{E})$ is acyclic, it is called an acyclic orientation. For each $u \in V$, let $d_{\vec{G}}(u)$ denote the number of edges in $\vec{G}$ directed towards u. The sequence $(d_{\vec{G}}(v_1), d_{\vec{G}}(v_2), \ldots, d_{\vec{G}}(v_n))$ is called the in-degree sequence of the orientation $\vec{G}$. We assume that the four basic arithmetic and comparison operations between two integer numbers can be done in constant time. Answer the following questions.

(1) Show that for two distinct acyclic orientations of G, their in-degree sequences are distinct.

(2) Design a linear time algorithm to determine whether there exists an acyclic orientation $\vec{G}$ of G with

$$(d_{\vec{G}}(v_1), d_{\vec{G}}(v_2), \ldots, d_{\vec{G}}(v_n)) = (h_1, h_2, \ldots, h_n)$$

or not for a given nonnegative integer sequence $(h_1, h_2, \ldots, h_n)$.

(3) Design a linear time algorithm to find an acyclic orientation $\vec{G}$ that minimizes $\max\{d_{\vec{G}}(v_i) \mid i = 1, 2, \ldots, n\}$.

(4) Design a linear time algorithm to find an acyclic orientation $\vec{G}$ that maximizes $\sum_{i=1}^{n} c_i d_{\vec{G}}(v_i)$ for a given integer sequence $(c_1, c_2, \ldots, c_n)$.