数理情報学専攻 修士課程入学試験問題

Department of Mathematical Informatics

Graduate School Entrance Examination Problem Booklet

専門科目 数理情報学

Specialized Subject: Mathematical Informatics

2024年8月19日（月） 10:00 – 13:00

August 19, 2024 (Monday) 10:00 – 13:00

5問出題，３問解答 / Answer 3 out of the 5 problems

注 意 事 項 / Instructions

(1) 試験開始の合図まで，この問題冊子を開かないこと．
Do not open this booklet until the starting signal is given.

(2) 本冊子に落丁，乱丁，印刷不鮮明の箇所などがあった場合には申し出ること．
Notify the proctor if there are missing or incorrect pages in your booklet.

(3) 本冊子には第1問から第5問まであり，日本文は4頁から13頁，英文は14頁から23頁である．5問のうち3問を日本語ないし英語で解答すること．
Five problems appear on pages 4–13 in Japanese and pages 14–23 in English in this booklet. Answer 3 problems in Japanese or English.

(4) 答案用紙3枚が渡される．1問ごとに必ず1枚の答案用紙を使用すること．止むを得ぬときは答案用紙の裏面を使用してもよい．
Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(5) 各答案用紙の指定された箇所に，受験番号およびその用紙で解答する問題番号を忘れずに記入すること．氏名は書いてはならない．
Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(6) 草稿用紙は本冊子から切り離さないこと．
Do not separate a draft sheet from the booklet.

(7) 解答に関係のない記号，符号などを記入した答案は無効とする．
Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(8) 答案用紙および問題冊子は持ち帰らないこと．
Leave the answer sheets and this booklet in the examination room.

受験番号 Examinee number	No.

上欄に受験番号を記入すること．

Fill in your examinee number.

選択した問題番号 Problem numbers			

上欄に選択した３つの問題番号を記入すること．

Fill in the three selected problem numbers.

Problem 1

Let d be an integer satisfying $d \geq 2$. For real symmetric matrices X, Y of size d, denote $X \succeq Y$ if $X - Y$ is positive semidefinite and denote $X \succ Y$ if $X - Y$ is positive definite. The sum of the diagonal entries of X is denoted by $\mathrm{tr}(X)$. Let O denote the square zero matrix of size d.

For each of the following (1)–(5) for real symmetric matrices A, B of size d, provide a proof if it is true, and a counterexample if it is false.

(1) If $A \succeq O$ and $B \succeq O$ hold, then $\mathrm{tr}(AB) \geq 0$ holds.

(2) If $A \succeq O$, $B \succeq O$, and $\mathrm{tr}(AB) = 0$ hold, then $A = O$ or $B = O$ holds.

(3) If $\mathrm{tr}(AC) \geq 0$ holds for any real positive semidefinite matrix C of size d, then $A \succeq O$ holds.

(4) If $A \succeq B \succeq O$ holds, then $A^2 \succeq B^2$ holds.

(5) If $A \succeq B \succ O$ holds, then $B^{-1} \succeq A^{-1}$ holds.

Problem 2

Consider the following initial-boundary problem of a partial differential equation with respect to the real-valued function $u(t, x)$.

$$\begin{cases} \dfrac{\partial u}{\partial t} = u - u^3 + \dfrac{\partial^2 u}{\partial x^2} & (0 < x < L,\ t > 0), \\ \dfrac{\partial u}{\partial x}(t, 0) = \dfrac{\partial u}{\partial x}(t, L) = 0 & (t > 0), \\ u(0, x) = f(x) & (0 \leq x \leq L). \end{cases} \tag{A}$$

Assume that L is a positive real number, and $f(x)$ is a continuous function satisfying $\sup\limits_{0\leq x\leq L} |f(x)| < 1$. Answer the following questions.

(1) Let $H(u) = \displaystyle\int_0^L \left\{ \frac{u^4}{4} - \frac{u^2}{2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right\} \mathrm{d}x$. Show that this is monotonically non-increasing in $t > 0$.

(2) Consider the initial-boundary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & (0 < x < L,\ t > 0), \\ \dfrac{\partial u}{\partial x}(t, 0) = \dfrac{\partial u}{\partial x}(t, L) = 0 & (t > 0), \\ u(0, x) = f(x) & (0 \leq x \leq L) \end{cases} \tag{B}$$

as a simplified version of the equation (A). For this problem, consider the case where the solution is written in the form

$$u(t, x) = \sum_{k=0}^{\infty} b_k \exp(c_k t) \cos\left(\frac{k\pi}{L} x\right),$$

where b_k, c_k $(k = 0, 1, 2, \ldots)$ are constants independent of x, t. Find b_k and c_k. Also, find the limit of the solution as $t \to \infty$.

(3) Consider the following ordinary differential equation with respect to a real-valued function $v(t)$ $(t \geq 0)$ depending only on t,

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} = v - v^3 & (t > 0), \\ v(0) = a. \end{cases} \tag{C}$$

Find the solution of the equation (C) when the initial value a satisfies $|a| < 1$. Then, find the limit of the solution as $t \to \infty$.

(4) The equation (A) is a mathematical model describing phase separation in systems where two substances are mixed together (phase separation means that the two substances are separated into their areas). In this model, $u(t, x)$ describes a ratio of mixing the two substances at time t and position x. Answer the following two questions about how the equation (A) is supposed to model phase separation.

(i) List main points on how the modeling of phase separation is affected by the asymptotic behaviors of the equations (B) and (C), the monotonically non-increasing property of $H(u)$, etc.

(ii) Show conceptual pictures of the evolution of the solution when started with sufficiently small initial data, i.e., when $f(x)$ satisfies $|f(x)| \ll 1$.

Problem 3

Let $\mathbb{R}$ be the set of all real numbers. For $a, b \in \mathbb{R}$ $(a < b)$, let $\mathcal{F}_{a,b}$ be the set of all real-valued continuous functions f defined on the open interval (a, b) that satisfy the following conditions:

- $\inf_{x \in (a,b)} f(x) > 0$,
- $\int_a^b f(x) \mathrm{d}x = 1$.

For $f \in \mathcal{F}_{a,b}$, define the mean of f by $\int_a^b x f(x) \mathrm{d}x$, and define the median of f by $z \in (a, b)$ such that $\int_a^z f(x) \mathrm{d}x = \int_z^b f(x) \mathrm{d}x = \frac{1}{2}$. Answer the following questions.

(1) Define a function $g \in \mathcal{F}_{0,\frac{5}{3}}$ as follows:

$$g(x) = \begin{cases} \frac{3}{2}x + \frac{1}{2} & \left(x \in \left(0, \frac{1}{3}\right]\right), \\ -\frac{3}{2}x + \frac{3}{2} & \left(x \in \left(\frac{1}{3}, \frac{2}{3}\right]\right), \\ \frac{1}{2} & \left(x \in \left(\frac{2}{3}, \frac{5}{3}\right)\right). \end{cases}$$

Find the mean and median of g.

In the following, let a, b be real numbers such that $a < b$, and let $f \in \mathcal{F}_{a,b}$.

(2) Denote the inverse function of the function $u = F(x) = \int_a^x f(\xi) \mathrm{d}\xi$ $(a < x < b)$ by $x = Q(u)$ $(0 < u < 1)$. Show that the mean of f is given by $\int_0^1 Q(u) \mathrm{d}u$.

(3) Show that the mean of f is strictly larger than the median of f, if f is strictly decreasing on the interval (a, b). Here, we say that f is strictly decreasing if it satisfies $f(x) > f(y)$ for $a < x < y < b$.

(4) For $c, d \in \mathbb{R}$ $(a < c < d < b)$, define $f_{cd} \in \mathcal{F}_{c,d}$ by

$$f_{cd}(x) = \frac{f(x)}{\int_c^d f(\xi) \mathrm{d}\xi}.$$

Suppose that f is differentiable on the interval (a, b) and the derivative f' of f is continuous on (a, b). Show that f is strictly decreasing on (a, b), if the mean of f_{cd} is strictly larger than the median of f_{cd} for any $c, d \in \mathbb{R}$ $(a < c < d < b)$.

Problem 4

Let n be a positive integer, and let $0 < p < 1$. Answer the following questions.

(1) For a positive real number λ, a random variable X is said to obey the Poisson distribution with mean λ if the equation

$$\Pr(X = k) = \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}$$

is satisfied for any non-negative integer k. For positive real numbers λ_1 and λ_2, let X_1 and X_2 be independent random variables obeying the Poisson distributions with mean λ_1 and λ_2, respectively. Then, show that $X_1 + X_2$ obeys the Poisson distribution with mean $\lambda_1 + \lambda_2$.

(2) Show that

$$\left|\Pr\left(\sum_{j=1}^{n}\xi_j \le K\right) - \Pr\left(\sum_{j=1}^{n}\eta_j \le K\right)\right| \le \sum_{j=1}^{n}\Pr(\xi_j \ne \eta_j)$$

for any real number K and any random variables $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$, which are not necessarily independent. You may use the fact that

$$\Pr\left(\bigcup_{j=1}^{n} A_j\right) \le \sum_{j=1}^{n}\Pr(A_j)$$

holds for any events $A_1, \ldots, A_n$.

(3) Let U be a random variable obeying the uniform distribution $\mathrm{U}(0,1)$, i.e.,

$$\Pr(U \in (a,b)) = b - a$$

for $0 \le a < b \le 1$. For $\ell = 0, 1, 2, \ldots$, let

$$q_\ell = \mathrm{e}^{-p}\sum_{i=0}^{\ell}\frac{p^i}{i!}.$$

Let ξ and η be random variables defined by

$$\xi = \begin{cases} 0 & (\text{if } U \le 1-p), \\ 1 & (\text{if } U > 1-p), \end{cases}$$

$$\eta = \min\{\ell \mid \ell \text{ is a non-negative integer, } U \le q_\ell\}.$$

Here, define $\eta = 0$ if $\{\ell \mid \ell \text{ is a non-negative integer, } U \le q_\ell\}$ is the empty set. Then, show that $\Pr(\xi \ne \eta) \le p^2$. You may use the fact that $1 - \mathrm{e}^{-x} \le x$ holds for any real number x.

(4) Let $\mu = np$. Let Y be a random variable obeying the binomial distribution $\mathrm{B}(n, p)$, i.e.,

$$\Pr(Y = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

for $k = 0, 1, \ldots, n$. Then, show that

$$\left| \Pr(Y \leq K) - \mathrm{e}^{-\mu} \sum_{i=0}^{K} \frac{\mu^i}{i!} \right| \leq \frac{\mu^2}{n}$$

for any $K = 0, 1, \ldots, n$.

Problem 5

Let $\mathbb{Z}$ be the set of all integers. For a matrix X, the (i, j)-th entry of X is denoted by $X[i, j]$.

Let n be a positive integer, and z_{ij} $(i = 1, \ldots, n, j = 1, \ldots, n)$ be integers. For an indeterminant x, a square matrix A of size n is defined by

$$A[i, j] = x^{z_{ij}} \qquad (i = 1, \ldots, n,\ j = 1, \ldots, n),$$

and for $k = 0, 1, 2, \ldots,$ define $S_k = \sum_{\ell=0}^{k} A^\ell$, where A^0 means the identity matrix of size n. Let $d_k(i, j)$ be the maximum degree among the non-zero coefficient terms of $S_k[i, j]$ in x, and define a square matrix D_k of size n by $D_k[i, j] = d_k(i, j)$. Answer the following questions.

(1) Provide a polynomial-time reduction of the problem of computing D_n for given $z_{ij} \leq 0$ $(i = 1, \ldots, n,\ j = 1, \ldots, n)$ to the shortest path problem. Here, the shortest path problem is the problem of computing a path between two vertices such that the sum of the weights along the path is minimized for a given edge-weighted directed graph.

(2) For given $z_{ij} \in \mathbb{Z}$ $(i = 1, \ldots, n,\ j = 1, \ldots, n)$, show that the following assertions are equivalent.

- Each entry of D_k converges to a finite value as $k \to \infty$.
- $D_n[i, i] \leq 0$ holds for all $i = 1, \ldots, n$.

(3) Provide a polynomial-time algorithm for deciding whether each entry of D_k converges to a finite value as $k \to \infty$ for given $z_{ij} \in \mathbb{Z}$ $(i = 1, \ldots, n,\ j = 1, \ldots, n)$.

(4) Suppose that, for given $s_{ij}, t_{ij} \in \mathbb{Z}$ $(i = 1, \ldots, n,\ j = 1, \ldots, n)$, each z_{ij} is written as a function

$$z_{ij} = s_{ij} + t_{ij}\lambda$$

in $\lambda \in \mathbb{Z}$. Provide a polynomial-time algorithm for deciding whether

$$\Lambda = \{\lambda \in \mathbb{Z} \mid \text{each entry of } D_k \text{ converges to a finite value as } k \to \infty\}$$

is the empty set.