数理情報学専攻 修士課程入学試験問題
Department of Mathematical Informatics
Graduate School Entrance Examination Problem Booklet
専門科目 数理情報学
Specialized Subject: Mathematical Informatics
2023 年 8 月 21 日（月） 10:00 – 13:00
August 21, 2023 (Monday) 10:00 – 13:00
5 問出題，3 問解答 / Answer 3 out of the 5 problems

注意事項 / Instructions
(1) 試験開始の合図まで、この問題冊子を開かないこと。
Do not open this booklet until the starting signal is given.
(2) 本冊子に落丁、乱丁、印刷不鮮明の箇所などがあった場合には申し出ること。
Notify the proctor if there are missing or incorrect pages in your booklet.
(3) 本冊子には第 1 問から第 5 問まであり、日本文は 4 頁から 13 頁、英文は 14 頁から 23 頁
である。5 問のうち 3 問を日本語ないし英語で解答すること。
Five problems appear on pages 4–13 in Japanese and pages 14–23 in English in this
booklet. Answer 3 problems in Japanese or English.
(4) 答用紙 3 枚が渡される。1 問ごとに必ず1枚の答用紙を使用すること。止むを得ぬとき
Three answer sheets will be given. Use one sheet per problem. If necessary, you may
ときは答用紙の裏面を使用してもよい。
use the back of the sheet.
(5) 各答用紙の指定された箇所に、受験番号およびその用紙で解答する問題番号を忘れずに
Fill in the examinee number and the problem number in the designated place of each
記入すること。氏名は書いてはならない。
answer sheet. Do not put your name.
(6) 草稿用紙は本冊子から切り離さないこと。
Do not separate a draft sheet from the booklet.
(7) 解答に関係のない記号、符号などを記入した答えは無効とする。
Any answer sheet with marks or symbols unrelated to the answer will be invalid.
(8) 答用紙および問題冊子は持ち帰らないこと。
Leave the answer sheets and this booklet in the examination room.

<table>
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<th>受験番号</th>
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上欄に受験番号を記入すること。
Fill in your examinee number.

<table>
<thead>
<tr>
<th>選択した問題番号</th>
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上欄に選択した 3 つの問題番号を記入すること。
Fill in the three selected problem numbers.
Problem 1

For a matrix $A \in \mathbb{R}^{d \times m}$, let $a_{i,j}$ be its $(i,j)$-th component, $A^\top$ be its transpose, and let $\|A\|_F = \sqrt{\sum_{i=1}^d \sum_{j=1}^m a_{i,j}^2}$. For a square matrix $A \in \mathbb{R}^{d \times d}$, its trace is $\text{tr}A = \sum_{i=1}^d a_{i,i}$. Let $I$ be the $d \times d$ identity matrix.

Suppose $d < m$. For given matrices $X, Y \in \mathbb{R}^{d \times m}$, let $\text{OPT}(X,Y)$ be the set of optimal solutions $P$ of the following optimization problem:

$$\min_{P \in \mathbb{R}^{d \times d}} \|PX - Y\|_F^2 \text{ subject to } P^\top P = I. \quad (*)$$

Answer the following questions.

1. Let $a_j$ and $b_j$ be the $j$-th column vectors of matrices $A$ and $B$, respectively, and let $\|a_j\|_2$ be the Euclidean norm of $a_j$. Find a pair of matrices $X, Y \in \mathbb{R}^{d \times m}$ such that $\text{OPT}(X,Y)$ equals the set of optimal solutions $P$ of the following optimization problem with given matrices $A, B \in \mathbb{R}^{d \times m}$ and positive real values $w_1, \ldots, w_m$:

$$\min_{P \in \mathbb{R}^{d \times d}} \sum_{j=1}^m w_j \|Pa_j - b_j\|_2^2 \text{ subject to } P^\top P = I.$$

2. Show that, for matrices $X, Y \in \mathbb{R}^{d \times m}$, $\text{OPT}(X,Y)$ equals the set of optimal solutions $P$ of the following optimization problem:

$$\min_{P \in \mathbb{R}^{d \times d}} \text{tr}(PXY^\top) \text{ subject to } P^\top P = I.$$

3. For matrices $X, Y \in \mathbb{R}^{d \times m}$, the singular value decomposition of the matrix $XY^\top$ is denoted by $XY^\top = USV^\top$. Give an optimal solution $P \in \text{OPT}(X,Y)$ of the optimization problem $(*)$ in terms of matrices among $X, Y, U, \Sigma, V$. 
Problem 2

Consider the minimization problem of a sufficiently smooth convex function $f(x)$ defined on $\mathbb{R}^d$. We assume the existence of an optimal solution $x^*$. We denote the gradient of $f(x)$ with respect to $x = (x_1, \ldots, x_d)^{\top} \in \mathbb{R}^d$ by $\nabla f(x) = \left( \frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_d}(x) \right)^{\top}$. We denote the Euclidean norm of $x \in \mathbb{R}^d$ by $\|x\|_2$. Answer the following questions.

(1) Consider the system of differential equations

\[
\begin{align*}
\frac{d}{dt}x(t) &= \frac{2}{t+1} (v(t) - x(t)) \\
\frac{d}{dt}v(t) &= -\frac{t+1}{2} \nabla f(x(t))
\end{align*}
\]

for $t \geq 0$. We assume that the initial conditions $x(0)$ and $v(0)$ are given. Let $E(t) = (t+1)^2(f(x(t)) - f(x^*)) + 2\|v(t) - x^*\|_2^2$. Show $\frac{d}{dt}E(t) \leq 0$ holds. Also show that there exists a constant $C$ independent of $t$ and $f(x(t)) - f(x^*) \leq C/t^2$ holds for every $t > 0$.

(2) As a discretization of the system of differential equations (1), consider

\[
\begin{align*}
\delta^+x^{(k)} &= \frac{2hk + h + 2}{(hk+1)^2} (v^{(k+1)} - x^{(k+1)}) \\
\delta^+v^{(k)} &= -\frac{2hk + h + 2}{4} \nabla f(x^{(k+1)})
\end{align*}
\]

for $k = 0, 1, 2, \ldots$, where $\delta^+$ is the operator that is defined for any scalar or vector sequence $\{y^{(k)}\}$ by $\delta^+y^{(k)} = (y^{(k+1)} - y^{(k)})/h$ for some constant $h > 0$. We set $x^{(0)} = x(0)$, $v^{(0)} = v(0)$ and assume that (2) has a solution. Let $E^{(k)} = (hk+1)^2(f(x^{(k)}) - f(x^*)) + 2\|v^{(k)} - x^*\|_2^2$. Show

$$\delta^+\|v^{(k)} - x^*\|_2^2 = 2(v^{(k+1)} - x^*)^\top (\delta^+v^{(k)}) - h\|\delta^+v^{(k)}\|_2^2,$$

and prove $\delta^+E^{(k)} \leq 0$. You may use the identity $\delta^+(a^{(k)}b^{(k)}) = (a^{(k+1)}b^{(k+1)} - a^{(k)}b^{(k)})/h = (\delta^+a^{(k)})b^{(k+1)} + a^{(k)}(\delta^+b^{(k)})$ that holds for any scalar sequences $\{a^{(k)}\}, \{b^{(k)}\}$ without proving it.

(3) We assume (2) has a solution. Show that there exists a constant $C'$ independent of $k$ and $f(x^{(k)}) - f(x^*) \leq C'/k^2$ holds for $k = 1, 2, \ldots$. 

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Problem 3

Let \( \mathbb{R} \) be the set of all the real numbers and let \( \mathbb{C} \) be the set of all the complex numbers. Let \( i \) be the imaginary unit and let \( e \) be the base of the natural logarithm. For a function \( g : \mathbb{R} \to \mathbb{C} \) with period \( 2\pi \), define its norm \( \|g\| \) by

\[
\|g\| := \sup_{\theta \in [0, 2\pi]} |g(\theta)|.
\]

For a positive integer \( n \), define

\[
\mathcal{T}_n := \{ t : \mathbb{R} \to \mathbb{C} \mid t(\theta) = P(e^{i\theta}) \text{ holds for some polynomial } P \text{ of degree } n \text{ with complex coefficients} \}.
\]

Let \( f : \mathbb{R} \to \mathbb{C} \) be a continuous function with period \( 2\pi \), and assume that there exists \( t_n \in \mathcal{T}_n \) such that

\[
\|f - t_n\| \leq \frac{c}{n^3}
\]

for each positive integer \( n \). Here \( c > 0 \) is a constant that is independent of \( n \). Answer the following questions.

1. For each positive integer \( n \), show that

\[
\|t_{n+1} - t_n\| \leq \frac{2c}{n^3}
\]

holds.

2. We write the derivative of \( t \in \mathcal{T}_n \) as \( t' \). Show that the sequence of functions \( \{t'_n\} \) uniformly converges on \( [0, 2\pi] \) to a continuous function \( s : \mathbb{R} \to \mathbb{C} \) with period \( 2\pi \). You may use the following two facts without proving them.

- Let \( C[0, 2\pi] \) be the set of all the complex-valued continuous functions on \( [0, 2\pi] \). Then, the normed space \( (C[0, 2\pi], \| \cdot \|) \) is complete.
- For each positive integer \( n \) and any \( t \in \mathcal{T}_n \),

\[
\|t'\| \leq n\|t\|
\]

holds.

3. Show that \( f \) is differentiable on \( \mathbb{R} \).
Problem 4

The exponential distribution $\text{Exp}(\lambda)$ for $\lambda > 0$ is the probability distribution on nonnegative real numbers with the probability density function

$$p(x) = \lambda \exp(-\lambda x).$$

Let $X_1, X_2, \ldots$ be a sequence of independent random variables obeying $\text{Exp}(\lambda)$. Answer the following questions.

(1) For $c > 0$, find the conditional probability density function of $X_1 - c$ given $X_1 \geq c$.

(2) Let $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. Find the probability density functions of $Y$ and $Z - Y$.

(3) Find the probability density function of $X_1 + \cdots + X_n$.

(4) For $a > 0$, let $N$ be the random variable defined by $X_1 + \cdots + X_N \leq a < X_1 + \cdots + X_{N+1}$. Let $N = 0$ if $a < X_1$. Find the maximum likelihood estimate $\lambda(N)$ of $\lambda$ based on $N$. Also, find the mean and the variance of $\lambda(N)$. 
Problem 5

Let $\mathbb{N}$ be the set of natural numbers (positive integers). Consider the following problem of assigning jobs to machines. For natural numbers $n$ and $m$, let $W = \{w_1, w_2, \ldots, w_n\}$ denote the set of jobs and $V = \{v_1, v_2, \ldots, v_m\}$ denote the set of machines. For each job $w_i \in W$, let $\ell_{w_i} \in \mathbb{N}$ be the processing time for job $w_i$ and $V_{w_i} \subseteq V$ the set of machines that can process job $w_i$. A mapping $\pi : W \rightarrow V$ is called a feasible assignment if

$$\pi(w_i) \in V_{w_i}$$

for each $w_i \in W$. Given a feasible assignment $\pi$, we define the load $T(\pi, v_j)$ of machine $v_j \in V$ as

$$T(\pi, v_j) = \sum_{w_i \in \pi^{-1}(v_j)} \ell_{w_i},$$

and the makespan $T(\pi)$ under $\pi$ as

$$T(\pi) = \max_{v_j \in V} T(\pi, v_j).$$

The minimum makespan problem is the problem of finding a feasible assignment $\pi$ that minimizes the makespan $T(\pi)$ over all feasible assignments. Answer the following questions.

1. Suppose that all the processing times $\ell_{w_i}$ of jobs are identical, i.e., $\ell_{w_i} = \ell_{w_j}$ for every pair of jobs $w_i, w_j \in W$. Provide a polynomial-time algorithm for the minimum makespan problem.

2. Consider the following greedy algorithm.

   1: (Initialization:) Let $\pi$ be a mapping $\pi : \emptyset \rightarrow V$ over the empty set.
   2: for $i = 1, 2, \ldots, n$ do
   3:     Choose an arbitrary machine $v^*$ from $\arg\min_{v_j \in V_{w_i}} T(\pi, v_j)$.
   4:     Extend $\pi$ to a mapping over $\{w_1, \ldots, w_i\}$ by $\pi(w_i) = v^*$.
   5: end for
   6: Output $\pi$.

Suppose that $V_{w_i} = V$ for every $w_i \in W$. Let $\pi_{\text{ALG}}$ be the output of the greedy algorithm and $T^*$ be the optimal makespan. Show that $T(\pi_{\text{ALG}}) \leq 2T^*$. 

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