数理情報学専攻 修士課程入学試験問題

Department of Mathematical Informatics

Graduate School Entrance Examination Problem Booklet

専門科目 数理情報学

Specialized Subject: Mathematical Informatics

2022年8月22日（月）　10:00 – 13:00

August 22, 2022 (Monday)　10:00 – 13:00

５問出題，３問解答 / Answer 3 out of the 5 problems

注　意　事　項 / Instructions

(1) 試験開始の合図まで，この問題冊子を開かないこと．
Do not open this booklet until the starting signal is given.

(2) 本冊子に落丁，乱丁，印刷不鮮明の箇所などがあった場合には申し出ること．
Notify the proctor if there are missing or incorrect pages in your booklet.

(3) 本冊子には第１問から第５問まであり，日本文は４頁から13頁，英文は14頁から23頁である．５問のうち３問を日本語ないし英語で解答すること．
Five problems appear on pages 4–13 in Japanese and pages 14–23 in English in this booklet. Answer 3 problems in Japanese or English.

(4) 答案用紙３枚が渡される．１問ごとに必ず１枚の答案用紙を使用すること．止むを得ぬときは答案用紙の裏面を使用してもよい．
Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(5) 各答案用紙の指定された箇所に，受験番号およびその用紙で解答する問題番号を忘れずに記入すること．氏名は書いてはならない．
Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(6) 草稿用紙は本冊子から切り離さないこと．
Do not separate a draft sheet from the booklet.

(7) 解答に関係のない記号，符号などを記入した答案は無効とする．
Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(8) 答案用紙および問題冊子は持ち帰らないこと．
Leave the answer sheets and this booklet in the examination room.

受験番号 Examinee number	No.

上欄に受験番号を記入すること．

Fill in your examinee number.

選択した問題番号 Problem numbers			

上欄に選択した３つの問題番号を記入すること．

Fill in the three selected problem numbers.

Problem 1

For finite real number sequences $A = \{a_i\}_{i=1}^{m}, B = \{b_j\}_{j=1}^{n}$ with positive integers m, n, define

$$f(A,B) = \ln\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\frac{1}{n}\sum_{j=1}^{n}\mathrm{e}^{-|a_i-b_j|}}\right),$$

where ln designates the natural logarithm. Answer the following questions.

(1) Show that $f(A,B) \geq 0$ holds, and describe a necessary and sufficient condition for A and B to satisfy $f(A,B) = 0$.

(2) Show that

$$f(A,C) \leq f(A,B) + f(B,C)$$

holds for any nonempty finite real number sequences A, B, C.

(3) For any real number s, define $A_m(s) = \{s + \frac{i}{m}\}_{i=1}^{m}$, $B_n = \{\frac{j}{n}\}_{j=1}^{n}$, and

$$g(s) = \lim_{m\to\infty}\lim_{n\to\infty} f(A_m(s), B_n).$$

Derive an explicit form of a function $h(z)$ that satisfies

$$g(s) = \ln\left(\int_{s}^{1+s}\frac{1}{h(z)}\mathrm{d}z\right).$$

(4) Find a real number s that minimizes $g(s)$.

Problem 2

Let u_i be the ith component of a vector u in the n dimensional real vector space $\mathbb{R}^n$. A vector $u \in \mathbb{R}^n$ is called a positive vector if it satisfies $u_i > 0$ for all $i = 1, \ldots, n$. The set of all positive vectors in $\mathbb{R}^n$ is denoted by $\mathcal{X}_n$. For a vector $a \in \mathbb{R}^n$, let $\mathrm{diag}(a_1, \ldots, a_n)$ denote the diagonal matrix whose ith diagonal component is a_i for $i = 1, \ldots, n$. For a matrix M, its transpose is denoted by $M^\top$.

For a real nonsingular $n \times n$ matrix $A = (a_{ij})$ and a vector $v \in \mathbb{R}^n$, a function $F_i(x)$ of $x \in \mathcal{X}_n$ is defined by

$$F_i(x) = x_i \left(v_i + \sum_{j=1}^{n} a_{ij} x_j \right) \qquad (i = 1, \ldots, n).$$

Assume that there exists a positive vector $x^* \in \mathcal{X}_n$ satisfying $F_i(x^*) = 0$ $(i = 1, \ldots, n)$. We consider a system of differential equations

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = F_i(x(t)) \qquad (t \geq 0,\ i = 1, \ldots, n) \qquad (*)$$

for $x(t) \in \mathcal{X}_n$. Answer the following questions.

(1) For a vector $c \in \mathcal{X}_n$, let $L(x)$ be a function of $x \in \mathcal{X}_n$ defined by

$$L(x) = \sum_{i=1}^{n} c_i \left[x_i^* \log \frac{x_i^*}{x_i} - x_i^* + x_i \right].$$

For the solution $x(t)$ of Equation $(*)$ with an initial state $x(0) = x' \in \mathcal{X}_n$, we define $\dot{L}(x')$ by $\dot{L}(x') = \left. \frac{\mathrm{d}L(x(t))}{\mathrm{d}t} \right|_{t=0}$. In addition, let $C = \mathrm{diag}(c_1, \ldots, c_n)$. Show that $\dot{L}(x') < 0$ holds for arbitrary $x' \in \mathcal{X}_n \setminus \{x^*\}$ if and only if the symmetric matrix $CA + A^\top C$ is negative definite.

(2) For a vector $w \in \mathcal{X}_n$, let $H_w(z)$ be a function of $z \in \mathbb{R}^n$ defined by

$$H_w(z) = \frac{1}{2} \sum_{i=1}^{n} w_i z_i^2.$$

We denote the gradient of $H_w(z)$ at z by $\nabla H_w(z) = \left(\frac{\partial H_w}{\partial z_1}(z), \ldots, \frac{\partial H_w}{\partial z_n}(z) \right)^\top$. For $z(t) = x(t) - x^*$ defined with the solution $x(t)$ of Equation $(*)$, obtain a matrix-valued function $G(z)$ that satisfies

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = G(z(t))\, \nabla H_w(z(t)).$$

The function $G(z)$ should be written in terms of A, $W = \mathrm{diag}(w_1, \ldots, w_n)$, $X^* = \mathrm{diag}(x_1^*, \ldots, x_n^*)$, and $Z = \mathrm{diag}(z_1, \ldots, z_n)$.

(3) Suppose that there exists a positive vector $c \in \mathcal{X}_n$ such that the symmetric matrix $CA + A^{\top}C$ is negative definite for $C = \mathrm{diag}(c_1, \dots, c_n)$. Find a vector $w \in \mathcal{X}_n$ satisfying the following statement.

> There exists an open neighborhood $U \subset \mathbb{R}^n$ of $0 \in \mathbb{R}^n$ such that $z(t) \in U \setminus \{0\}$ implies $\dfrac{\mathrm{d}H_w(z(t))}{\mathrm{d}t} < 0$.

Problem 3

Let $\mathbb{C}$ be the set of all complex numbers and i denote the imaginary unit. For a real number $r > 1$, let D_r be the disk in $\mathbb{C}$ defined by $D_r = \{z \in \mathbb{C} \mid |z| < r\}$. For a holomorphic function $f : D_r \to \mathbb{C}$ on D_r, denote by $\|f\|$ the supremum norm of f, i.e., $\|f\| = \sup_{z \in D_r} |f(z)|$, and define $I(f)$ and $I_N(f)$ by

$$I(f) = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \quad \text{and} \quad I_N(f) = \frac{2\pi}{N} \sum_{k=1}^{N} f(\mathrm{e}^{2\pi \mathrm{i} k/N}),$$

respectively, where N is a positive integer. Answer the following questions.

(1) For a real number $R > 0$, let $\Gamma(R) \subset \mathbb{C}$ be the circle centered at 0 and of radius R with positive (counterclockwise) orientation. Show that

$$I(f) = \oint_{\Gamma(1)} \frac{-\mathrm{i}}{z} f(z)\,\mathrm{d}z$$

holds for a holomorphic function $f : D_r \to \mathbb{C}$ on D_r.

(2) For a holomorphic function $f : D_r \to \mathbb{C}$ on D_r, let

$$g_N[f](z) = \frac{-\mathrm{i}\, z^{N-1}}{z^N - 1} f(z).$$

Find all poles of $g_N[f]$ on D_r and calculate the residue of $g_N[f]$ at each pole.

(3) Assume that a holomorphic function $f : D_r \to \mathbb{C}$ on D_r satisfies $\|f\| < \infty$. Show that

$$|I(f) - I_N(f)| \leq \frac{2\pi \|f\|}{r^N - 1} \qquad (*)$$

holds.

(4) Show that the constant 2π in the right-hand side of Inequality $(*)$ is optimal, i.e.,

$$\limsup_{N \to \infty} \left(r^N \sup_f \frac{|I(f) - I_N(f)|}{\|f\|} \right) = 2\pi$$

holds, where $\sup_f$ designates the supremum over all holomorphic functions $f : D_r \to \mathbb{C}$ on D_r with $\|f\| < \infty$.

Problem 4

Let X be a random variable obeying the normal distribution with mean 0 and variance 1. Let $f(y)$ denote the probability density function of a random variable

$$Y = \frac{1}{X^2}.$$

Let i be the imaginary unit, and let $\mathbb{R}$ be the set of real numbers. The expectation of a random variable Z is denoted by $\mathbb{E}[Z]$. Answer the following questions.

(1) Find $f(y)$.

(2) Denote the Laplace transform of the function $f(y)$ by $L(u) = \int_0^\infty \mathrm{e}^{-uy} f(y)\mathrm{d}y$ $(u \geq 0)$. Then, show that the equation

$$\frac{\mathrm{d}L(u)}{\mathrm{d}u} = -\frac{1}{\sqrt{2u}} L(u) \quad (u > 0)$$

holds.

(3) Denote the characteristic function of Y by $\varphi(u) = \mathbb{E}[\mathrm{e}^{\mathrm{i}uY}]$ $(u \in \mathbb{R})$. Find $\varphi(u)$.

(4) Let $Y_1, \ldots, Y_n$ be independently identically distributed random variables obeying the probability density function $f(y)$. Show that

$$\frac{1}{n^2}(Y_1 + \cdots + Y_n)$$

converges in distribution (converges in law) as $n \to \infty$, and find the probability density function of the limit distribution.

Problem 5

A ternary representation of a natural number (positive integer) d is a sequence of integers $(d_0, d_1, \ldots, d_n)$ satisfying

$$d = \sum_{i=0}^{n} d_i 2^i, \quad d_i \in \{-1, 0, 1\} \quad (i = 0, 1, \ldots, n-1), \quad d_n = 1.$$

A ternary representation $(d_0, d_1, \ldots, d_n)$ of d satisfying

$$d_{i-1} d_i = 0$$

for all integers i with $1 \leq i \leq n$ is called a sparse ternary representation. Answer the following questions.

(1) For a natural number n, find the maximum integer L_n that can be represented by a sparse ternary representation $(d_0, d_1, \ldots, d_n)$.

(2) Show that the sparse ternary representation of an arbitrary natural number d is uniquely determined.

(3) Design an $\mathrm{O}(\log d)$-time algorithm for converting the binary representation of a natural number d to its sparse ternary representation.

(4) For a sequence of integers $(a_0, a_1, \ldots, a_n)$, let $w(a_0, a_1, \ldots, a_n)$ denote the number of nonzero integers a_i. Show that

$$w(d_0^*, d_1^*, \ldots, d_m^*) \leq w(d_0, d_1, \ldots, d_n)$$

holds for the sparse ternary representation $(d_0^*, d_1^*, \ldots, d_m^*)$ and any ternary representation $(d_0, d_1, \ldots, d_n)$ of a natural number d.

(5) For a natural number n, let X_n be a random variable that obeys the discrete uniform distribution over the set $\{z \in \mathbb{N} \mid z \leq L_n\}$. A random variable Y_n is defined by $Y_n = w(d_0^*, d_1^*, \ldots, d_m^*)$ by using the sparse ternary representation $(d_0^*, d_1^*, \ldots, d_m^*)$ of X_n. Show that

$$\lim_{n \to \infty} \frac{\mathbb{E}[Y_n]}{n} = \frac{1}{3}$$

holds. Here, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{E}[Y_n]$ denotes the expected value of Y_n.