数理情報学専攻 修士課程入学試験問題

Department of Mathematical Informatics

Graduate School Entrance Examination Problem Booklet

専門科目 数理情報学

Specialized Subject: Mathematical Informatics

2019年8月20日（火）　10:00 – 13:00

August 20, 2019 (Tuesday)　10:00 – 13:00

5問出題，3問解答 / Answer 3 out of the 5 problems

注　意　事　項 / Instructions

(1) 試験開始の合図まで，この問題冊子を開かないこと．
Do not open this booklet until the starting signal is given.

(2) 本冊子に落丁，乱丁，印刷不鮮明の箇所などがあった場合には申し出ること．
Notify the proctor if there are missing or incorrect pages in your booklet.

(3) 本冊子には第1問から第5問まであり，日本文は4頁から13頁，英文は14頁から23頁である．5問のうち3問を日本語ないし英語で解答すること．
Five problems appear on pages 4–13 in Japanese and pages 14–23 in English in this booklet. Answer 3 problems in Japanese or English.

(4) 答案用紙3枚が渡される．1問ごとに必ず1枚の答案用紙を使用すること．止むを得ぬときは答案用紙の裏面を使用してもよい．
Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(5) 各答案用紙の指定された箇所に，受験番号およびその用紙で解答する問題番号を忘れずに記入すること．氏名は書いてはならない．
Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(6) 草稿用紙は本冊子から切り離さないこと．
Do not separate a draft sheet from the booklet.

(7) 解答に関係のない記号，符号などを記入した答案は無効とする．
Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(8) 答案用紙および問題冊子は持ち帰らないこと．
Leave the answer sheets and this booklet in the examination room.

受験番号 Examinee number	No.

上欄に受験番号を記入すること．

Fill in your examinee number.

選択した問題番号 Problem numbers			

上欄に選択した3つの問題番号を記入すること．

Fill in the three selected problem numbers.

Problem 1

Let n be a positive integer. Let A be a real square matrix of size n, and let B be a real symmetric positive-definite matrix of size n. Answer the following questions.

(1) Show that there exists a unique real square matrix C of size n satisfying

$$BC + CB = A.$$

In the following, this matrix C is denoted by $C_{A,B}$.

(2) Show that $BC_{A,B} = C_{A,B}B$ if and only if $AB = BA$.

Problem 2

The lifetime of an organism is distributed according to the exponential distribution with mean μ. We want to estimate μ by observing lifetimes of n individuals of the organism. However, due to a restriction of the experiment, observation is not available in the time interval $[0, a]$ immediately after the birth, and only the fact of death is observed if the organism has died in the interval. Here a is a positive constant.

For $i = 1, \dots, n$, denote the lifetime of the i-th individual by X_i. They are distributed according to the exponential distribution with mean μ and are mutually independent. The probability density function of the exponential distribution with mean μ is $f(x; \mu) = (1/\mu)\mathrm{e}^{-x/\mu}$ $(x > 0)$. Define an observation Y_i by

$$Y_i = \begin{cases} a & (\text{if } X_i \leq a), \\ X_i & (\text{if } X_i > a). \end{cases}$$

Answer the following questions.

(1) Denote the expectation of Y_1 by $g(\mu)$. Find $g(\mu)$.

(2) Let $\bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i$. Prove that if $\bar{Y} > a$, there exists a unique $\hat{\mu}$ such that $g(\hat{\mu}) = \bar{Y}$.

(3) Let M denote the number of i's such that $Y_i = a$. For $0 \leq m \leq n - 1$ and $b > a$, find a function $h(m, y; \mu)$ satisfying

$$\mathrm{P}(M = m, \bar{Y} \leq b) = \int_a^b h(m, y; \mu)\mathrm{d}y.$$

(4) Prove that if $m < n$ and $y > a$, then there exists a unique μ that maximizes $h(m, y; \mu)$.

Problem 3

Let K be a field, and let n be a positive integer. Let $K(x_1, x_2, \ldots, x_n)$ be the field of rational functions over K with n variables, and let $L = K[x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}]$ be the subring of $K(x_1, x_2, \ldots, x_n)$ generated by K and $x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$. Let $R = K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be the polynomial ring over K with $2n$ variables. Answer the following questions.

(1) For an element $p \in R$, let $\varphi(p)$ denote the element of L obtained by substituting x_i^{-1} into each variable y_i $(i = 1, \ldots, n)$ in p. This map $\varphi : R \to L$ is a ring homomorphism. Show that for an ideal J of L, $\varphi^{-1}(J)$ is an ideal of R.

(2) For $1 \leq i \leq n$, let $g_i = x_i y_i - 1$. Let

$$R' = \left\{ r \in R \;\middle|\; \begin{array}{l} \text{for } 1 \leq i \leq n, \text{ every monomial in } r \\ \text{does not involve } x_i \text{ and } y_i \text{ simultaneously} \end{array} \right\}.$$

Show that for an arbitrary element $p \in R$, there exist $h_1, \ldots, h_n \in R$ and $r \in R'$ such that $p = h_1 g_1 + \cdots + h_n g_n + r$.

(3) Let I denote the ideal of R generated by $g_1, \ldots, g_n$. Show that $\ker \varphi = I$ and that L is isomorphic to the quotient ring R/I.

Problem 4

Consider the differential equation

$$\frac{\partial}{\partial t}x(n,t) = x(n-1,t) + x(n+1,t) - 2\,x(n,t) \qquad (*)$$

for a function $x(n,t)$ of an integer n and a real number $t \geq 0$. Assume that the function $x(n,t)$ satisfies

$$x(n+N,t) = x(n,t) \qquad (**)$$

for any integer n, where N is an integer larger than or equal to 3. Furthermore, let $e(m,n) = \exp\left(\mathrm{i}\frac{2\pi mn}{N}\right)$ for integers m and n, where i is the imaginary unit. Answer the following questions.

(1) Let $f_m(t)$ be a function of a real number $t \geq 0$ for an integer m with $f_m(0) = c_m$, where c_m is a complex number. Assume that the function of the form $x(n,t) = e(m,n)\,f_m(t)$ satisfies differential equation $(*)$ and condition $(**)$. Find $f_m(t)$.

(2) Let $(g_0, \ldots, g_{N-1})$ be an N-dimensional complex vector. Under the initial condition $x(n,0) = g_n \quad (n = 0, 1, \ldots, N-1)$, find the solution of differential equation $(*)$ with condition $(**)$.

(3) Find $\lim_{t\to\infty} x(n,t)$ for the solution $x(n,t)$ found in (2).

Problem 5

Consider arrays of rational numbers. Let $A[i]$ denote the i-th element of an array A. For an array A of length n, a monotone non-decreasing array B of length n that minimizes

$$\sum_{i=1}^{n}(A[i]-B[i])^2$$

is called an approximate array of A. Here an array B is said to be monotone non-decreasing if $B[1] \leq B[2] \leq \cdots \leq B[n]$. Answer the following questions.

(1) Let A' be the array made by appending an element at the tail of array A of length n. That is, $A'[i] = A[i]$ $(1 \leq i \leq n)$. Let B and B' be approximate arrays of A and A', respectively.

(1-1) Prove that if $A'[n+1] \geq B[n]$, then $B'[i] = B[i]$ $(1 \leq i \leq n)$ and $B'[n+1] = A'[n+1]$ hold.

(1-2) Assume that $B[1] = B[2] = \cdots = B[n]$ and $A'[n+1] < B[n]$. Show that $B'[1] = B'[2] = \cdots = B'[n+1]$ and $B'[n+1] < B[n]$, and find the value of $B'[n+1]$.

(2) Give a polynomial-time algorithm for finding an approximate array of an array A. We assume that the four basic arithmetic operations between two rational numbers can be done in constant time.