Department of Mathematical Informatics

Graduate School Entrance Examination
Problem Booklet

Specialized Subject: Mathematical Informatics

August 21, 2018 (Tuesday)   10:00 – 13:00

Answer 3 out of the 5 problems

Please note:

(1) Do not open this booklet until the starting signal is given.

(2) Notify the proctor if there are missing or incorrect pages in your booklet.

(3) Answer in Japanese or English.

(4) Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(5) Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(6) Do not separate a draft sheet from the booklet.

(7) Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(8) Leave the answer sheets and this booklet in the examination room.

Examinee number	No.

Fill in your examinee number.

Problem numbers			

Fill in numbers of the three answered problems.

Problem 1

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be real square matrices of order n satisfying

$$a_{ij} \geq 0 \ (i,j = 1,2,\ldots,n), \quad \sum_{i=1}^{n} a_{ij} = 1 \ (j = 1,2,\ldots,n), \quad B = \alpha A + \frac{1-\alpha}{n}\mathbb{1}\mathbb{1}^\top,$$

where $\top$ stands for transpose, α is a real number satisfying $0 < \alpha < 1$, and $\mathbb{1} = (1,\ldots,1)^\top \in \mathbb{R}^n$ denotes the vector with all components equal to 1. For $v = (v_1,\ldots,v_n)^\top \in \mathbb{R}^n$, let $||v||_1 := \sum_{i=1}^{n} |v_i|$. Answer the following questions.

(1) Obtain the maximum absolute value of the eigenvalues of the matrix A.

(2) Show that there exists a vector $x \in \mathbb{R}^n$ that has a nonnegative value in each component and satisfies

$$Bx = x, \quad \mathbb{1}^\top x = 1.$$

You may use the following fact:

"Every continuous map from a nonempty compact convex set in $\mathbb{R}^n$ to itself has a fixed point."

(3) Show that for a vector $q = (q_1,\ldots,q_n)^\top \in \mathbb{R}^n$ satisfying $\mathbb{1}^\top q = 0$, it holds that

$$\left| \sum_{j=1}^{n} b_{ij} q_j \right| \leq \sum_{j=1}^{n} b_{ij} |q_j| - \frac{1-\alpha}{n} \|q\|_1 \quad (i = 1,2,\ldots,n).$$

(4) Show that for a positive integer N, a vector $x \in \mathbb{R}^n$ with the conditions in (2) satisfies

$$\left\| B^N \frac{\mathbb{1}}{n} - x \right\|_1 \leq \alpha^N \left\| \frac{\mathbb{1}}{n} - x \right\|_1.$$

Problem 2

Let $\mathbb{R}$ be the set of real numbers. Let X_i $(i = 1, 2, \ldots, k)$ be k independently identically distributed random variables such that each X_i is an $\mathbb{R}^2$-valued random variable obeying the bivariate normal distribution with mean $\mathbf{0} = (0, 0)^\top$ and covariance $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Here $\top$ stands for transpose. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(x) := \max\{x, 0\}$ $(x \in \mathbb{R})$. For $u \in \mathbb{R}^2$, let

$$g_u(x_1, \ldots, x_k) := \frac{1}{k} \sum_{j=1}^{k} \varphi(u^\top x_j) \quad (x_j \in \mathbb{R}^2,\ j = 1, 2, \ldots, k).$$

Then, for a fixed nonzero vector $u^* \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, we consider the problem of minimizing

$$L(u) = \mathrm{E}_X \left[\left(g_u(X_1, \ldots, X_k) - g_{u^*}(X_1, \ldots, X_k) \right)^2 \right]$$

with respect to $u \in \mathbb{R}^2$, where E_X denotes the expectation with respect to the random variable $X = (X_1, \ldots, X_k)$. Answer the following questions.

(1) Calculate $\mathrm{E}_X[\varphi(u^\top X_i)\varphi(u^\top X_j)]$ $(1 \le i, j \le k)$, and write it by using u.

In the following, consider the case of $k = 1$. For $u \neq \mathbf{0}$, let $0 \le \theta \le \pi$ be the angle between u and u^*, namely, $\theta := \cos^{-1}\left(\dfrac{u^\top u^*}{\sqrt{(u^\top u)(u^{*\top} u^*)}} \right)$.

(2) Obtain the value of the following integral:

$$\int_0^\infty \left(\int_{\theta - \pi/2}^{\pi/2} r^3 \cos(\psi) \cos(\theta - \psi) \frac{1}{2\pi} \exp\left(-\frac{r^2}{2} \right) \mathrm{d}\psi \right) \mathrm{d}r.$$

(3) Calculate $L(u)$, and write it by using $\|u\|$, $\|u^*\|$ and θ.

(4) Enumerate all local optimal solutions for the optimization problem of minimizing $L(u)$ over $u \in \mathbb{R}^2$. Moreover, enumerate all global optimal solutions.

Problem 3

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$, and the norm of x is defined by $\|x\| := \sqrt{\langle x, x \rangle}$. Answer the following questions.

(1) For a nonempty closed set $K \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, show the following:

(1-1) There exists $y \in K$ satisfying $\|x - y\| = \inf_{z \in K} \|x - z\|$.

(1-2) If K is convex, then such a point y is uniquely determined.

(1-3) If K is convex and x is not contained in K, then there exist $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ such that

$$\begin{aligned} \langle c, x \rangle &> d, \\ \langle c, z \rangle &\leq d \quad (z \in K). \end{aligned}$$

(2) For a subset $\mathcal{A}$ of $\mathbb{R}^n$, let $C(\mathcal{A}) \subseteq \mathbb{R}^n$ denote the set of all nonnegative combinations of elements in $\mathcal{A}$. Namely, $C(\mathcal{A})$ is the set consisting of points x represented as $x = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$ for elements $a_1, a_2, \ldots, a_m$ in $\mathcal{A}$ $(m \geq 1)$ and nonnegative reals $\lambda_1, \lambda_2, \ldots, \lambda_m$.

(2-1) For $\mathcal{A} \subseteq \mathbb{R}^n$, show that the following holds:

$$C(\mathcal{A}) = \bigcup_{\mathcal{B}} C(\mathcal{B}),$$

where the union is taken over all linearly independent subsets $\mathcal{B}$ of $\mathcal{A}$.

(2-2) Show that if $\mathcal{A} \subseteq \mathbb{R}^n$ is a finite set, then $C(\mathcal{A})$ is a closed set.

(3) For an $n \times m$ real matrix $A \in \mathbb{R}^{n \times m}$ and n-dimensional vector $x \in \mathbb{R}^n$, consider the following two properties (P) and (Q):

(P) There exists $\lambda \in \mathbb{R}^m$ satisfying $A\lambda = x$ and $\lambda \geq 0$.

(Q) There exists $c \in \mathbb{R}^n$ satisfying $c^\top A \leq 0$ and $c^\top x > 0$.

Here $\top$ stands for transpose, and for a vector u, notation $u \geq 0$ $(u \leq 0)$ means that each component of u is nonnegative (nonpositive). Show the following:

(3-1) It never happens that both of (P) and (Q) hold.

(3-2) (P) or (Q) holds.

Problem 4

For a complex square matrix $X \in \mathbb{C}^{n\times n}$ of order n, let $\mathrm{e}^X := \sum_{k=0}^{\infty} \frac{1}{k!} X^k$, where X^0 is the identity matrix I of order n. Furthermore, the norm of an n-dimensional complex vector $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ is defined by $\|x\| := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$, where $\top$ stands for transpose. Also, the norm of a matrix $X \in \mathbb{C}^{n\times n}$ is defined by $\|X\| := \sup_{x \in \mathbb{C}^n,\ \|x\|=1} \|Xx\|$. Then, for any $X, Y \in \mathbb{C}^{n\times n}$, it holds that $\|X+Y\| \le \|X\|+\|Y\|$ and $\|XY\| \le \|X\|\|Y\|$. Answer the following questions.

(1) For $X \in \mathbb{C}^{n\times n}$, prove the following inequalities:

$$\|\mathrm{e}^X\| \le \mathrm{e}^{\|X\|}, \quad \|\mathrm{e}^X - \mathrm{I}\| \le \|X\|\, \mathrm{e}^{\|X\|}, \quad \|\mathrm{e}^X - \mathrm{I} - X\| \le \|X\|^2\, \mathrm{e}^{\|X\|}.$$

In the following, let $A, B \in \mathbb{C}^{n\times n}$, and let m be an integer greater than or equal to 1. Define $P := \mathrm{e}^{(A+B)/m}$ and $Q := \mathrm{e}^{A/m}\mathrm{e}^{B/m}$.

(2) Show the following inequality:

$$\|P^m - Q^m\| \le m\, \|P - Q\|\, \mathrm{e}^{\frac{m-1}{m}(\|A\|+\|B\|)}.$$

You may use the relation $P^m - Q^m = \sum_{i=0}^{m-1} P^i (P-Q) Q^{m-1-i}$ without a proof.

(3) Show the following equality:

$$P - Q = g\left(\frac{1}{m}(A+B)\right) - g\left(\frac{1}{m}A\right) - g\left(\frac{1}{m}B\right) - f\left(\frac{1}{m}A\right) f\left(\frac{1}{m}B\right),$$

where $f(X) := \mathrm{e}^X - \mathrm{I}$ and $g(X) := \mathrm{e}^X - \mathrm{I} - X$ for $X \in \mathbb{C}^{n\times n}$.

(4) Show the following inequality:

$$\|P^m - Q^m\| \le \frac{2}{m} \left(\|A\| + \|B\|\right)^2\, \mathrm{e}^{\|A\|+\|B\|}.$$

(5) Consider the initial value problem of the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = (A+B)\, x(t), \quad x(0) = v$$

with respect to an n-dimensional complex vector-valued function $x(t)$. Let the sequence $\tilde{x}^0, \tilde{x}^1, \ldots, \tilde{x}^{2m}$ of n-dimensional complex vectors be determined by $\tilde{x}^0 = v$ and

$$\tilde{x}^{k+1} = \begin{cases} \mathrm{e}^{B/m}\, \tilde{x}^k & (k \text{ is even}), \\ \mathrm{e}^{A/m}\, \tilde{x}^k & (k \text{ is odd}), \end{cases} \qquad (k = 0, 1, \ldots, 2m-1).$$

Show that for any real number α with $0 < \alpha < 1$,

$$\lim_{m\to\infty} m^\alpha \left\| x(1) - \tilde{x}^{2m} \right\| = 0.$$

Problem 5

Consider a tree T with n nodes. Define a centroid of T as any node c of T satisfying the following condition:

> "Each of trees $T_1, T_2, \ldots$ obtained from T by removing the node c has at most $n/2$ nodes."

Answer the following questions, where we assume that any number can be represented in O(1) space.

(1) Let $T_1, T_2, \ldots$ be trees obtained from the tree T by removing a node v. Show that if v is not a centroid of T, then centroids of T exist in exactly one of the trees $T_1, T_2, \ldots$.

From T, we create another rooted tree R with n nodes. There is a one-to-one correspondence between the nodes in T and those in R. Let $\bar{v}$ denote the node in R corresponding to a node v in T. The rooted tree $R = \mathrm{R}(T)$ is obtained by the following recursive algorithm:

- If T consists only of one node v, then $\mathrm{R}(T)$ is defined as the rooted tree consisting only of one node $\bar{v}$.
- Otherwise, choose a centroid c of T arbitrarily, and for trees $T_1, T_2, \ldots$ obtained from T by removing c, create $R_1 = \mathrm{R}(T_1), R_2 = \mathrm{R}(T_2), \ldots$ recursively. Then create a new node $\bar{c}$, and define $\mathrm{R}(T)$ as the rooted tree whose root is $\bar{c}$ and its children are the roots of $R_1, R_2, \ldots$.

Figure 1 shows the procedure to create a rooted tree R from a tree T.

(2) Show that the height of R is $\mathrm{O}(\log n)$.

(3) Consider a node $\bar{v}$ of R. Let R_i and R_j be distinct subtrees which are children of $\bar{v}$, and let $\bar{v}_i$ and $\bar{v}_j$ be nodes in R_i and R_j, respectively. Show that a simple path in T from v_i to v_j passes the node v, and moreover, the path passes only nodes of T corresponding to nodes of the subtree rooted at $\bar{v}$ in R.

(4) For nodes $\bar{v}, \bar{w}$ in R, define $\mathrm{lca}(\bar{v}, \bar{w})$ as the common ancestor of $\bar{v}$ and $\bar{w}$ whose path from the root has the maximum length. Show that $\mathrm{lca}(\bar{v}, \bar{w})$ can be computed in $\mathrm{O}(\log n)$ time.

(5) Show that by adding a data structure of $\mathrm{O}(n \log n)$ space to T in advance, the length of a simple path between two nodes of T can be computed in $\mathrm{O}(\log n)$ time.

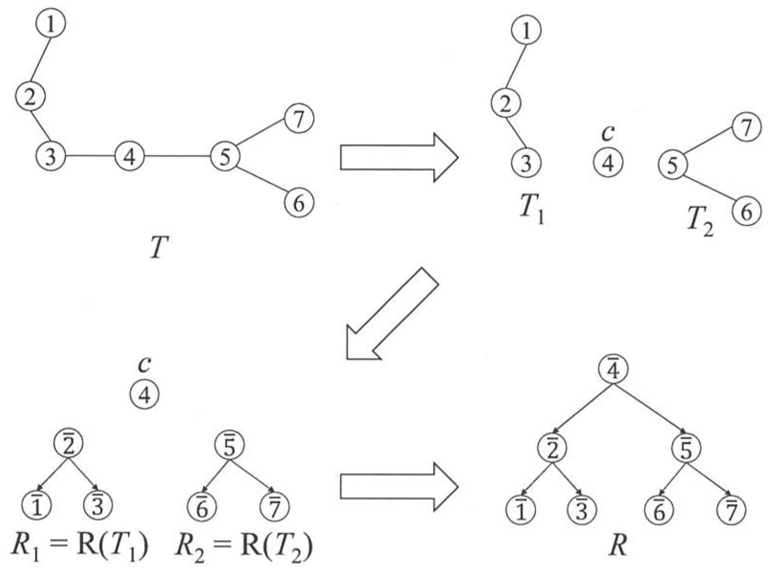

Figure 1. (Top left) a tree T. (Top right) trees T_1, T_2 obtained from T by removing a centroid c. (Bottom left) the rooted trees $R_1 = \mathrm{R}(T_1)$, $R_2 = \mathrm{R}(T_2)$ created from T_1, T_2. (Bottom right) the rooted tree $R = \mathrm{R}(T)$.