数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成２９年８月２２日（火）　１０：００～１３：００

５問出題，３問解答

This booklet is an informal English translation of the original examination booklet. Answer in Japanese or English.

Answer three out of the five problems.

Please note:

(1) Do not open this booklet until the starting signal is given.

(2) Notify the proctor if there are missing or incorrect pages in your booklet.

(3) Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(4) Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(5) Do not separate a draft sheet from the booklet.

(6) Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(7) Leave the answer sheets and this booklet in the examination room.

Examinee number	No.

Fill in your examinee number.

Problem numbers			

Fill in numbers of the three answered problems.

Problem 1

Let $\mathbb{C}$ be the field of complex numbers. Let A be an n by n matrix over $\mathbb{C}$. An invariant subspace of A is a subspace U of $\mathbb{C}^n$ such that $AU \subseteq U$. Let $\mathcal{S}_A$ denote the set of all invariant subspaces of A. A partial order $\preceq$ on $\mathcal{S}_A$ is defined as the inclusion relation $\subseteq$, and $\mathcal{S}_A$ is regarded as a partially ordered set.

Answer the following questions. Refer to the remark below for lattice and Hasse diagram.

(1) (1-1) Show that $\mathcal{S}_A$ is a lattice.

(1-2) Show that $\mathcal{S}_A$ and $\mathcal{S}_{P^{-1}AP}$ are isomorphic, as a partially ordered set, for any nonsingular matrix P.

(1-3) Show that $\mathcal{S}_A = \mathcal{S}_{A+\alpha \mathrm{I}}$ holds for any complex number α, where I is the identity matrix.

(2) Draw the Hasse diagram of $\mathcal{S}_A$ when A is each of the following matrices:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -2 & -1 & -2 \\ 4 & 3 & 2 \\ 4 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 1 \\ -2 & -1 & 0 \\ -3 & -1 & -1 \end{pmatrix}.$$

(3) Explain for what kind of matrix A the corresponding $\mathcal{S}_A$ consists of a finite number of elements, and explain how the Hasse diagram of such $\mathcal{S}_A$ looks like.

(Remark). A partially ordered set $\mathcal{L}$ with partial order $\preceq$ is called a lattice if for every pair $x, y \in \mathcal{L}$ the following two properties hold:

- There exists a unique maximal element in the set $\{u \in \mathcal{L} \mid x \succeq u \preceq y\}$.
- There exists a unique minimal element in the set $\{u \in \mathcal{L} \mid x \preceq u \succeq y\}$.

Also, the Hasse diagram of $\mathcal{L}$ is the directed graph obtained from $\mathcal{L}$ by making every element of $\mathcal{L}$ a vertex and adding an edge from x to y for every pair of distinct elements $x, y \in \mathcal{L}$ with the following property:

- $x \preceq y$, and $\{z \in \mathcal{L} \mid x \preceq z \preceq y\} = \{x, y\}$.

Problem 2

Let $\mathbb{R}$ be the set of real numbers. Suppose that P_1 and P_2 are probability distributions on $\mathbb{R}$. When P_1 and P_2 have probability density functions $p_1, p_2 : \mathbb{R} \to \mathbb{R}$, respectively, satisfying $0 < p_1(x)/p_2(x) < \infty$ $(\forall x \in \mathbb{R})$, the Kullback-Leibler divergence from P_2 to P_1 is defined by

$$D(P_1||P_2) = \int_{-\infty}^{\infty} p_1(x) \log \left(\frac{p_1(x)}{p_2(x)} \right) \mathrm{d}x.$$

The normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is denoted by $\mathrm{N}(\mu, \sigma^2)$. The expectation of a random variable Z is denoted by $\mathrm{E}[Z]$. For a cumulative distribution function Ψ, the function $\Psi^{-1} : (0,1) \to \mathbb{R}$ is defined by $\Psi^{-1}(t) = \inf\{x \in \mathbb{R} \mid \Psi(x) > t\}$. Answer the following questions.

(1) Obtain $D(P_1||P_2)$ when P_1 and P_2 are $\mathrm{N}(\mu_1, {\sigma_1}^2)$ and $\mathrm{N}(\mu_2, {\sigma_2}^2)$, respectively.

(2) Consider two random variables X and Y that have marginal distributions P_1 and P_2, respectively. Assume that X and Y have finite second moments and satisfy $P_1(X \geq 0) = P_2(Y \geq 0) = 1$.

(2-1) Let P_{XY} denote the joint distribution of X and Y. Show the following equality:

$$\mathrm{E}[X \cdot Y] = \int_0^{\infty} \int_0^{\infty} P_{XY}(\{X \geq x\} \cap \{Y \geq y\}) \mathrm{d}x \mathrm{d}y.$$

(2-2) Let F and G denote the cumulative distribution functions of the probability distributions P_1 and P_2, respectively. Let U be a random variable obeying the uniform distribution on the open interval $(0,1)$. Then, show that the inequality

$$\mathrm{E}[(X - Y)^2] \geq \mathrm{E}[(F^{-1}(U) - G^{-1}(U))^2]$$

holds.

(3) Using the cumulative distribution functions F, G of P_1, P_2 and a random variable U obeying the uniform distribution on the open interval $(0,1)$, define a distance between P_1 and P_2 as

$$W(P_1, P_2) = \sqrt{\mathrm{E}[(F^{-1}(U) - G^{-1}(U))^2]}.$$

Assume that P_1 and P_2 are $\mathrm{N}(\mu_1, {\sigma_1}^2)$ and $\mathrm{N}(\mu_2, {\sigma_2}^2)$, respectively. Show that

$$W(P_1, P_2)^2 = (\mu_1 - \mu_2)^2 + (\sigma_1 - \sigma_2)^2$$

holds. Moreover, show that

$$W(P_1, P_2)^2 \leq 2{\sigma_2}^2 D(P_1||P_2)$$

holds, and derive a necessary and sufficient condition for the equality to hold in this inequality.

Problem 3

Let n be a positive integer. Let $\mathbb{R}$ be the field of real numbers. For a square matrix $M \in \mathbb{R}^{n\times n}$, denote the sum of diagonal elements of M by $\mathrm{tr}(M)$ and the transpose of M by $M^\top$. For square matrices $M, N \in \mathbb{R}^{n\times n}$, let $\langle M, N\rangle := \mathrm{tr}(M^\top N)$. The sphere $\mathbb{S}^2$ is defined by $\mathbb{S}^2 := \{(\xi,\eta,\zeta)^\top \in \mathbb{R}^3 \mid \xi^2+\eta^2+\zeta^2 = 1\}$.

For $C = (c_{ij}) \in \mathbb{R}^{n\times n}$, consider the following optimization problem (P1) with variables $p_1, \ldots, p_n \in \mathbb{S}^2$:

$$\begin{array}{lll} \text{(P1)} & \text{Maximize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} p_i^\top p_j \\ & \text{subject to} & p_i \in \mathbb{S}^2 \quad (i = 1, \ldots, n). \end{array}$$

Answer the following questions.

(1) Show that the optimal value of (P1) is less than or equal to the optimal value of the following optimization problem (P2) with variable $X \in \mathbb{R}^{n\times n}$:

$$\begin{array}{lll} \text{(P2)} & \text{Maximize} & \langle C, X\rangle \\ & \text{subject to} & \text{each element on the diagonal of } X \text{ is } 1, \\ & & X \text{ is a symmetric positive semidefinite matrix.} \end{array}$$

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be the permutation matrix corresponding to the cyclic permutation $(1, 2, \ldots, n)$, that is,

$$a_{ij} = \begin{cases} 1 & (j - i \equiv 1 \bmod n), \\ 0 & (\text{otherwise}). \end{cases}$$

Then the equality

$$\sum_{k=0}^{n-1} A^{-k} X A^k = \sum_{\ell=0}^{n-1} \langle A^\ell, X\rangle A^\ell$$

holds for any square matrix $X \in \mathbb{R}^{n\times n}$.

In the following, suppose that C is written as $C = \sum_{k=0}^{n-1} d_k A^k$ with $d_k \in \mathbb{R}$ $(k = 0, \ldots, n-1)$.

(2) Show that, if X is an optimal solution of (P2), then $\frac{1}{n}\sum_{k=0}^{n-1} A^{-k} X A^k$ is also an optimal solution of (P2). In addition, show that the optimal value of (P2) coincides with the optimal value of the following optimization problem (P3) with variables $y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$ and $Y \in \mathbb{R}^{n\times n}$:

$$\begin{array}{lll} \text{(P3)} & \text{Maximize} & \langle C, Y\rangle \\ & \text{subject to} & Y = \displaystyle\sum_{k=0}^{n-1} y_k A^k, \\ & & \text{each element on the diagonal of } Y \text{ is } 1, \\ & & Y \text{ is a symmetric positive semidefinite matrix.} \end{array}$$

(3) Show that the optimal value of (P2) coincides with the optimal value of the following linear programming problem (P4) with variables $y_1, \ldots, y_{n-1} \in \mathbb{R}$:

$$
\begin{aligned}
\text{(P4)} \quad \text{Maximize} \quad & nd_0 + \sum_{i=1}^{n-1} nd_i y_i \\
\text{subject to} \quad & \sum_{i=1}^{n-1} y_i \cos \frac{2\pi ij}{n} \geq -1 \quad (j = 0, \ldots, n-1), \\
& y_j = y_{n-j} \qquad\qquad\qquad (j = 1, \ldots, n-1).
\end{aligned}
$$

(4) Obtain the optimal value and an optimal solution of (P1) for $n = 4$ and $(d_0, d_1, d_2, d_3) = (0, 3, -4, 3)$.

Problem 4

Let $\mathbb{R}$ be the field of real numbers. Let us consider the following ordinary differential equations regarding functions $\theta_1, \theta_2 : \mathbb{R} \to \mathbb{R}$:

$$(*) \begin{cases} \dfrac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} = f(\theta_1(t), \theta_2(t)) := K \sin(\theta_1(t) - \theta_2(t)) - \sin(\theta_1(t)), \\ \dfrac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} = g(\theta_1(t), \theta_2(t)) := K \sin(\theta_2(t) - \theta_1(t)) - \sin(\theta_2(t)), \end{cases}$$

where $K > \frac{1}{2}$. Answer the following questions.

(1) Obtain all the stationary solutions $(\theta_1(t), \theta_2(t)) = (\theta_1^*, \theta_2^*)$ of the ordinary differential equations $(*)$ in the ranges $0 \leq \theta_1^* < 2\pi$ and $0 \leq \theta_2^* < 2\pi$, where θ_1^* and θ_2^* are constants independent of t.

(2) Let us define a matrix J by

$$J(\theta_1, \theta_2) := \begin{pmatrix} \dfrac{\partial f(\theta_1, \theta_2)}{\partial \theta_1} & \dfrac{\partial f(\theta_1, \theta_2)}{\partial \theta_2} \\ \dfrac{\partial g(\theta_1, \theta_2)}{\partial \theta_1} & \dfrac{\partial g(\theta_1, \theta_2)}{\partial \theta_2} \end{pmatrix}.$$

A stationary solution $(\theta_1(t), \theta_2(t)) = (\theta_1^*, \theta_2^*)$ is stable if all the real parts of the eigenvalues of $J(\theta_1^*, \theta_2^*)$ are negative. Determine whether the stationary solutions depending on K among those obtained in (1) are stable or not.

(3) Show that there exists a function $V(\theta_1, \theta_2)$ such that

$$f(\theta_1, \theta_2) = -\frac{\partial V(\theta_1, \theta_2)}{\partial \theta_1}, \quad g(\theta_1, \theta_2) = -\frac{\partial V(\theta_1, \theta_2)}{\partial \theta_2}.$$

(4) Show that the ordinary differential equations $(*)$ have no periodic solution. Here, a solution $\boldsymbol{\theta}(t) := (\theta_1(t), \theta_2(t))$ is called periodic if there exists $T > 0$ such that $\boldsymbol{\theta}(t + T) = \boldsymbol{\theta}(t)$ and $\boldsymbol{\theta}(t + s) \neq \boldsymbol{\theta}(t)$ for any s with $0 < s < T$.

Problem 5

Let m, n be natural numbers with $m > n \geq 1$, and denote by $\gcd(m, n)$ the greatest common divisor of m and n. Let $\mathbb{Z}$ be the integer ring. Answer the following questions.

(1) Show that $m\mathbb{Z} + n\mathbb{Z}$ is a principal ideal of $\mathbb{Z}$ generated by $\gcd(m, n)$.

(2) Let r be the remainder of the division of m by n. Show that $\gcd(m, n) = \gcd(n, r)$ holds.

(3) Construct an algorithm that takes natural numbers m, n with $m > n \geq 1$ as an input and computes integers x, y satisfying $mx + ny = \gcd(m, n)$ by using $O(\log m)$ arithmetic operations over $\mathbb{Z}$. Here, the arithmetic operations over $\mathbb{Z}$ are operations to compute the addition, the subtraction, the multiplication, and the quotient and remainder, for given two integers.

(4) Show that the quotient ring $\mathbb{Z}/p\mathbb{Z}$ is a field if and only if p is prime.

(5) Compute the inverse of 822 in the multiplicative group $(\mathbb{Z}/2017\mathbb{Z})^*$ for prime 2017.