数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成２８年８月２３日（火）　１０：００～１３：００

５問出題，３問解答

This booklet is an informal English translation of the original examination booklet. Answer in Japanese or English.

Answer three out of the five problems.

Please note:

(1) Do not open this booklet until the starting signal is given.

(2) Notify the supervisor if there are missing or incorrect pages in your booklet.

(3) Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(4) Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(5) Do not separate a draft sheet from the booklet.

(6) Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(7) Leave the answer sheets and this booklet in the examination room.

Examinee number	No.

Fill in your examinee number.

Problem numbers			

Fill in numbers of the three answered problems.

Problem 1

Let $F : \mathbb{R} \to \mathbb{R}$ be a map defined by $F(x) = a^x$, where $\mathbb{R}$ denotes the set of real numbers, x is a real variable, a is a positive real constant, and a^x is a to the power x. The derivative of $F(x)$ is denoted by $f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$. When $F(\bar{x}) = \bar{x}$ and $|f(\bar{x})| < 1$, we call $\bar{x}$ a stable fixed point of F. Answer the following questions.

(1) Suppose that $\bar{x}$ is a stable fixed point of F. Show that there exists an open interval J including $\bar{x}$ such that $\lim_{n\to\infty} F^n(x) = \bar{x}$ for every $x \in J$, where $F^0(x) = x$ and $F^n(x) = F(F^{n-1}(x))$ for each natural number n.

(2) Find the range of values of a such that F admits a stable fixed point $\bar{x}$, and the corresponding range of values of $\bar{x}$.

Problem 2

Suppose that there are two coins A and B. The probability of getting a head when tossing A is θ_{A} $(0 < \theta_{\mathrm{A}} < 1)$. The probability of getting a head when tossing B is θ_{B} $(0 < \theta_{\mathrm{B}} < 1)$. A sequence of n coin tosses is generated according to the following rule. If the coin shows a head, then the same coin is used for the next toss. If the coin shows a tail, then the other coin is used for the next toss. One of the two coins A and B is picked at random with probability $1/2$ for the initial toss. Answer the following questions.

(1) Obtain the probability that the coin A is used on the nth toss.

(2) Let $H(n)$ be the expectation of the number of heads in the n tosses. Obtain

$$\lim_{n \to \infty} \frac{H(n)}{n}.$$

(3) Show that the value obtained in (2) is greater than $(\theta_{\mathrm{A}} + \theta_{\mathrm{B}})/2$ if $\theta_{\mathrm{A}} \neq \theta_{\mathrm{B}}$.

Problem 3

Let p be a constant satisfying $0 < p < 1$. For a real-valued random variable X, define

$$R[X] = \frac{1}{1-p}\int_p^1 F_X{}^{-1}(u)\mathrm{d}u,$$

provided that the distribution function $F_X : \mathbb{R} \to (0,1)$ of X has its inverse $F_X{}^{-1}$ and satisfies $\int_0^1 |F_X{}^{-1}(u)|\mathrm{d}u < \infty$, where $\mathbb{R}$ denotes the set of real numbers and $(0,1) = \{x \mid 0 < x < 1\}$. Answer the following questions.

Here the expectation of a random variable X is denoted by $\mathrm{E}[X]$. For an event A, let $\Pr(A)$ denote the probability that A occurs, and let I_A denote the random variable that takes 1 if A occurs and 0 otherwise.

(1) Let T be a random variable with the distribution function $F_T(t) = 1/(1 + \mathrm{e}^{-t})$. Obtain $R[T]$.

(2) Consider a random variable X for which $R[X]$ is definable, and let B be the event that $X \geq F_X{}^{-1}(p)$ holds. Show that

$$\Pr(B) = 1 - p, \quad R[X] = \frac{\mathrm{E}[X \cdot I_B]}{1-p}$$

hold. In addition, show that the inequality

$$\mathrm{E}[X \cdot I_A] \leq \mathrm{E}[X \cdot I_B]$$

holds for any event A satisfying $\Pr(A) = 1 - p$.

(3) For random variables X and Y that are not necessarily independent, show that the inequality

$$R[X + Y] \leq R[X] + R[Y]$$

holds, provided that all of $R[X], R[Y]$, and $R[X + Y]$ are definable.

Problem 4

For square matrices A, B of the same size, we define (A, B) and $[A, B]$ by

$$(A, B) = AB + BA,$$
$$[A, B] = AB - BA.$$

The transpose of a matrix A is denoted by $A^\top$, and a matrix A is said to be skew-symmetric if $A^\top = -A$. Consider the initial value problem with respect to a real square matrix $X(t)$:

$$(*) \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}X(t) = [(M, X(t)), X(t)], \\ X(0) = X_0, \end{cases}$$

where M is a skew-symmetric matrix of the same size as $X(t)$, and X_0 is a symmetric matrix. Answer the following questions.

(1) Show that the solution $X(t)$ of $(*)$ is symmetric for all $t \geq 0$.

Using the solution $X(t)$ of $(*)$, consider another differential equation with respect to a real square matrix $Q(t)$ of the same size as $X(t)$:

$$(**) \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}Q(t) = (M, X(t))Q(t), \\ Q(0) = I, \end{cases}$$

where I denotes the identity matrix.

(2) Show that the solution $X(t)$ of $(*)$ for $t \geq 0$ can be written as $X(t) = Q(t)X_0Q(t)^\top$, where $Q(t)$ is the solution of $(**)$.

(3) Show that the solution $X(t)$ of $(*)$ has the same eigenvalues as X_0 for all $t \geq 0$.

Next, consider a numerical solution method for the initial value problem $(*)$. Let Δt be the discretization width of t, and compute approximate solutions X_k, Q_k to $X(t)$, $Q(t)$ at $t = k\Delta t$ $(k = 1, 2, \ldots)$ by the recursion:

$$(\dagger) \begin{cases} \dfrac{Q_k - Q_{k-1}}{\Delta t} = (M, X_{k-1})\dfrac{Q_k + Q_{k-1}}{2}, \\ X_k = Q_k X_{k-1} {Q_k}^\top, \end{cases}$$

where $Q_0 = Q(0) = I$, and X_0 is the one given in $(*)$.

(4) Show that X_k, Q_k $(k = 1, 2, \ldots)$ satisfying $(\dagger)$ are uniquely determined and that X_k are symmetric matrices.

(5) Show that the approximate solutions X_k $(k = 1, 2, \ldots)$ have the same eigenvalues as X_0.

Problem 5

We consider a connected undirected graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Let d_i denote the number of edges incident to v_i. Here G is assumed to have neither self-loops nor multiple edges. Let $A = (a_{ij})$ and $L = (l_{ij})$ be $n \times n$ matrices defined by

$$a_{ij} = \begin{cases} 1 & (\text{if } \{v_i, v_j\} \in E), \\ 0 & (\text{otherwise}), \end{cases} \qquad l_{ij} = \begin{cases} -1 & (\text{if } \{v_i, v_j\} \in E), \\ d_i & (\text{if } i = j), \\ 0 & (\text{otherwise}). \end{cases}$$

Answer the following questions.

(1) What does the (i, j)-component of the kth power A^k of the matrix A represent?

(2) The distance between two vertices is defined to be the minimum number of edges in a path connecting them. Show that the distance between any pair of vertices is less than the number of distinct eigenvalues of A.

(3) Show that $\sum_{i=1}^{n} u_i = 0$ holds for an eigenvector $u = (u_i)$ corresponding to a nonzero eigenvalue of the matrix L.

(4) Show that all the eigenvalues of the matrix L are nonnegative real numbers.

(5) We define the function $V : \mathbb{R}^n \to \mathbb{R}$ by

$$V(x) = \frac{1}{2} \sum_{1 \leq i < j \leq n} a_{ij}(x_i - x_j)^2 \quad (x \in \mathbb{R}^n),$$

and consider a system of differential equations

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\left.\frac{\partial V(x)}{\partial x_i}\right|_{x=x(t)} \qquad (i = 1, 2, \ldots, n)$$

for $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. Obtain the limit $\bar{x} = \lim_{t\to\infty} x(t)$ of the solution $x(t)$ for an initial value $x(0) = (c_1, c_2, \ldots, c_n)$, and discuss the rate of the convergence.