数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成２７年８月２５日（火）　１０：００～１３：００

５問出題，３問解答

This booklet is an informal English translation of the original examination booklet. Answer in Japanese or English.

Answer three out of the five problems.

Please note:

(1) Do not open this booklet until the starting signal is given.

(2) Notify the supervisor if there are missing or incorrect pages in your booklet.

(3) Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(4) Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(5) Do not separate a draft sheet from the booklet.

(6) Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(7) Leave the answer sheets and this booklet in the examination room.

Examinee number	No.

Fill in your examinee number.

Problem numbers			

Fill in numbers of the three answered problems.

Problem 1

Let n be a positive integer. In this problem, matrices are supposed to be over the real numbers. For a square matrix M, denote the sum of diagonal elements of M by $\mathrm{tr}(M)$ and the transpose of M by $M^\top$. Answer the following questions.

(1) Let A be a symmetric positive definite matrix of order n. Show that there exists a unique symmetric positive definite matrix R such that $R^2 = A$. We denote such R by $\sqrt{A}$.

(2) Let B be a nonsingular matrix of order n. Show that an orthogonal matrix Q that maximizes $f(Q) = \mathrm{tr}(QB)$ satisfies

$$Q = \sqrt{B^\top B}^{-1} B^\top = B^\top \sqrt{BB^\top}^{-1}.$$

(3) Let G and H be symmetric positive definite matrices of order n. Find a square matrix L that minimizes

$$g(L) = \mathrm{tr}\{(I - L)G(I - L)^\top\}$$

subject to $LGL^\top = H$, where I denotes the identity matrix of order n.

Problem 2

Let N be a nonnegative integer. Suppose that $2N+1$ random variables $X_1, \ldots, X_{2N+1}$ are independently distributed according to the probability density function

$$f(x;\mu) = \frac{1}{2}\exp(-|x-\mu|) \qquad (x \in \mathbb{R})$$

with an unknown parameter $\mu \in \mathbb{R}$. Let $F(x;\mu)$ be the distribution function of this distribution. Answer the following questions.

(1) Obtain the expectation and the variance of the sample mean $\dfrac{1}{2N+1}\displaystyle\sum_{i=1}^{2N+1} X_i$.

(2) Obtain the maximum likelihood estimator $\widehat{\mu}$ of μ based on $X_1, \ldots, X_{2N+1}$.

(3) Show that the probability density function $h(x;\mu)$ of the probability distribution that $\widehat{\mu}$ follows is given by

$$h(x;\mu) = (2N+1)\binom{2N}{N} f(x;\mu)\{F(x;\mu)\}^N\{1-F(x;\mu)\}^N \qquad (x \in \mathbb{R}).$$

(4) Let V_{2N+1} be the variance of the maximum likelihood estimator $\widehat{\mu}$. Here V_{2N+1} does not depend on μ. Obtain the asymptotic variance

$$\lim_{N\to\infty}(2N+1)V_{2N+1}$$

of $\sqrt{2N+1}(\widehat{\mu}-\mu)$.

(5) Which is the better estimator for the parameter μ, the sample mean or the maximum likelihood estimator?

Problem 3

Consider a population of individuals within a certain area. Each individual takes one of three states S, I, and R. Let $S(t)$, $I(t)$, and $R(t)$ be nonnegative real numbers representing the population densities of individuals in the states S, I, and R at time $t \geq 0$, respectively. The time evolution of the densities follows the differential equations

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \mu(I(t) + R(t)) - \beta S(t)I(t), \\
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta S(t)I(t) - (\mu + \gamma)I(t), \\
\frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \gamma I(t) - \mu R(t),
\end{aligned}$$

where μ, β, and γ are positive parameters. For arbitrary nonnegative initial values, the solution preserves the nonnegativity for all $t \geq 0$. Answer the following questions.

(1) Show that the density of all the individuals, $S(I) + I(t) + R(t)$, is an invariant independent of time.

Let us denote the invariant by K. In the following, we eliminate $R(t)$ and consider the differential equations

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \mu K - \mu S(t) - \beta S(t)I(t), \\
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta S(t)I(t) - (\mu + \gamma)I(t).
\end{aligned}$$

(2) Find the stationary solution $(S(t), I(t)) = (S^\circ, I^\circ)$ that is independent of parameter values of μ, β, and γ.

(3) Derive the condition for the parameters μ, β, and γ to admit a unique stationary solution $(S(t), I(t)) = (S^*, I^*)$ satisfying $S^* > 0$ and $I^* > 0$. In addition, find the stationary solution (S^*, I^*).

In the following, we assume that this condition is satisfied.

(4) Show that the function

$$V(S, I) = S^* \left(\frac{S}{S^*} - 1 - \log \frac{S}{S^*} \right) + I^* \left(\frac{I}{I^*} - 1 - \log \frac{I}{I^*} \right)$$

defined on $D = \{(S, I) \mid S > 0, I > 0\}$ attains its minimum at $(S, I) = (S^*, I^*)$.

(5) Show that a solution $(S(t), I(t))$ satisfies

$$\left. \frac{\mathrm{d}V(S(t), I(t))}{\mathrm{d}t} \right|_{t=\tau} \leq 0$$

for $\tau \geq 0$ with $S(\tau) > 0$ and $I(\tau) > 0$.

(6) Show that the solution $(S(t), I(t))$ converges to (S^*, I^*) as $t \to \infty$ for arbitrary initial values $S(0) > 0$ and $I(0) > 0$.

Problem 4

Consider a population of particles that can generate the same type of particles as themselves. Let $S(t)$ be the set of the particles of the population at the tth generation, and $X(t)$ denote the cardinality of $S(t)$. For each generation, each particle $i \in S(t)$ generates ξ_i new particles and then disappears. Thereby, the population size at the $(t+1)$th generation is

$$X(t+1) = \sum_{i \in S(t)} \xi_i.$$

Suppose that each ξ_i follows the geometric distribution $\mathrm{Ge}(q)$ with parameter q $(0 < q < 1)$ mutually independently, and that ξ_i is also independent of the past history of the population size, $X(0), X(1), \ldots, X(t)$. Answer the following questions, assuming that $X(0) = 1$.

(1) If a random variable ξ follows the geometric distribution $\mathrm{Ge}(q)$, the probability that it takes value k is given by

$$\Pr(\xi = k) = q(1-q)^k \quad (k = 0, 1, \ldots).$$

Obtain the expectation $\mathrm{E}[\xi]$ and the probability generating function $G(z) = \mathrm{E}[z^{\xi}]$ of ξ.

(2) Obtain the expectation $\mu(t) = \mathrm{E}[X(t)]$ of the random variable $X(t)$.

(3) Obtain the conditional probability generating function $H_t(z) = \mathrm{E}[z^{X(t)} | X(t-1)]$ as a function of $G(z)$.

(4) Obtain the probability generating function $F_t(z) = \mathrm{E}[z^{X(t)}]$ of $X(t)$ by using G.

(5) Let $p(t)$ denote the probability $\Pr(X(t) = 0)$ that the population has been extinct by the tth generation. Express $p(t+1)$ as a function of $p(t)$.

(6) Obtain the extinction probability $P_{\mathrm{E}} = \lim_{t \to \infty} p(t)$ of the population.

Problem 5

For an array A of length n storing a real value in each entry, define

$$f(s,t) = \prod_{i=s}^{t} A[i] \quad (1 \leq s \leq t \leq n).$$

We assume that a real value can be stored in a unit word of memory. Answer the following questions.

(1) Let P be an array such that $P[i] = f(1,i)$ $(1 \leq i \leq n)$. Assuming that the array A does not contain 0 in its entry, design an algorithm for computing $f(s,t)$ in $O(1)$ time for any given s,t with the aid of P.

(2) Assuming that each entry of A is either 0 or 1, design an $O(n)$-space data structure for computing $f(s,t)$ in $O(1)$ time for any given s,t.

(3) Design an $O(n)$-space data structure for computing $f(s,t)$ in $O(1)$ time for any given s,t.

(4) Design a data structure that fulfills all the following requirements, and describe algorithms for (b) and (c).

(a) The data structure uses $O(n)$ space.

(b) One can compute $f(s,t)$ in $O(\log n)$ time for any given s,t.

(c) When an entry of A has changed, one can update the data structure in $O(\log n)$ time.