Convexity of Generalized Matching Games
（一般化マッチングゲームの凸性）

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1 Cooperative Games

The *cooperative game theory*, whose history goes back to the seminal work of von Neumann and Morgenstern [13], treats the situation in which the players can gain larger profits by cooperating to each other. The main conceptual problem of cooperative game theory is to find a "good way", referred to as a *solution concept*, to distribute the value to the players when all players cooperate.

From computational aspect, researchers are interested in algorithms related to the concepts of "good ways". This is the one of the main interests of *computational cooperative game theory* [3].

Let us formulate cooperative game mathematically. A *cooperative game* (V, ν) is given by a finite set V and a function $\nu\colon 2^V \to \mathbb{R}$ with $\nu(\emptyset) = 0$, where V is the set of n *players* and ν is the *characteristic function* such that $\nu(X)$ indicates the value if players $X \subseteq V$ form a coalition.

One of the most famous solution concepts is the *Shapley value* [9], which is explicitly defined by the formula

$$\sum_{X \subseteq V \setminus \{v\}} \frac{|X|!(|V|-|X|-1)!}{|V|!}(\nu(X \cup \{v\}) - \nu(X)). \quad (1)$$

and represent the fair distribution.

A *convex game* [10] is a class of games with several desirable properties. Formally, a cooperative game is *convex* if its characteristic function $\nu\colon 2^V \to \mathbb{R}$ is *supermodular*, i.e., the following inequality holds for all $X, Y \subseteq V$ [6]:

$$\nu(X) + \nu(Y) \leq \nu(X \cap Y) + \nu(X \cup Y). \quad (2)$$

2 Optimization Problem Games

The research of *optimization problem games* is started by Shapley and Shubik [10], who defined the *assignment game*. In this line of research, researchers treat games such that $\nu(X)$ is defined by the optimal value of a combinatorial optimization problem on the substructure X. In this thesis, we consider the following two games.

Hypergraph Matching Game [4, 5] Let $G = (V, E)$ be a hypergraph and $c : E \to \mathbb{R}_{>0}$ be a weight function. The player set of this game is V and $\nu(X)$ is given by the maximum weight of the hypergraph matching of $G[X]$. Here, the hypergraph matching is a subset M of hyperedges such that each vertex in V is contained in at most one hyperedge of M.

b-matching Game [2, 11] Let $G = (V, E)$ be a graph, $b : V \to \mathbb{Z}_{>0}$ be a budget function and $c : E \to \mathbb{R}_{>0}$ be a weight function. The player set of this game is V and $\nu(X)$ is given by the maximum weight of the b-matching of $G[X]$. Here, a b-matching of $G[X]$ is a vector $x \in \mathbb{Z}^E_{\geq 0}$ such that

$$\sum_{e \text{ is incident to } v} x(e) \leq b(v) \quad (3)$$

holds for all $v \in X$.

Several research investigate the convexity in optimization problem games. The polynomial-time checkable characterization is known for matroid minimum base game [8], minimum spanning tree game [7], and minimum coloring game and minimum vertex cover game. It is investigated the structure of the convex hypergraph matching game [4], although polynomial-time characterization is not given.

3 Our Contribution

In this thesis, we investigate computational problems and structures related to the convexity of the hypergraph matching game and the b-matching game. Our main contribution is a polynomial-time algorithm to solve the following problems.

Problem 1. *Given a hypergraph matching game, de-*

termine it is convex or not.

Problem 2. *Given a b-matching game, determine it is convex or not.*

For the both games, we establish a necessary and sufficient condition of the convexity and construct an algorithm for checking the condition efficiently.

For the hypergraph matching game, we prove that it is sufficient to check polynomial number of supermodular inequalities (2) to check the convexity of the game. To prove this, we use a structural lemma on "essential" hyperedges. For the b-matching game, we can state much about its structure for the convex case. Specifically, we show that the whole graph should be the comparability graph of a branching.

Computing the Shapley value already #P-hard to calculate for a matching game [1], which is a common case of the hypergraph matching game and the b-matching game. Here, we restrict the instance into convex games and give polynomial-time algorithms to the following problems.

Problem 3. *Given a convex hypergraph matching game, compute the Shapley value.*

Problem 4. *Given a convex b-matching game, compute the Shapley value.*

The algorithms heavily depend on our characterizations of the convexity. For both games,we observe that the marginal gain, $\nu(X \cup \{v\}) - \nu(X)$, takes polynomial number of different values. For each value, we count the number of the pair (X, v) with that value of marginal gain. For the hypergraph matching game, this number is computed by the double-counting argument with the Möbius inversion formula [12]. For the b-matching game, we explicitly write this number using the branching that defines the whole graph.

Since the fractional relaxation of the matching problems define different problems, they define different games. However, we prove that, in convex case they coincide; the fractional versions are convex if and only if the corresponding integral versions are convex.

Because the convexity is a strong condition, most instances of the hypergraph matching game are nonconvex. Therefore, we consider modifying the game by compensating the vertices not in the matching to make the game convex.

Problem 5. *Given a hypergraph matching game, compute the minimum total compensation to make the game convex.*

We prove that this problem is NP-hard in general and admits a 2-approximation algorithm if the whole graph is an antichain.

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