Problem 1

For a matrix denoted by M, we write $M_{i:j,k:l}$ for the submatrix formed from the rows i through j and the columns from k through l of the matrix. We also denote $M_{i:i,k:l}$ and $M_{i:j,k:k}$ by $M_{i,k:l}$ and $M_{i:j,k}$, respectively. (Note that they are row and column vectors, respectively.)

Assume below that A is an $n \times n$ nonsingular matrix which can be LU factorized. In addition, assume that a positive integer w exists such that all entries in the k-th sub- and super-diagonal of A are zero if $k > w$. Namely, if $|i - j| > w$, $A_{i,j}$ is zero.

We denote the LU factorization of A by $A = LU$, where L is a lower triangular matrix with unit diagonal elements, and U is an upper triangular matrix.

The following algorithm **P** computes L and U from the input A in-place in such a way that the strictly lower triangular part and the upper triangular part of A will be L and U, respectively. In other words, $A_{i,j}$ will be $L_{i,j}$ for $i > j$, and $A_{i,j}$ will be $U_{i,j}$ for $i \le j$, when it terminates.

Algorithm **P**

for($k = 1$; $k < n$; $k = k + 1$) {

$$A_{k+1:n,k} \leftarrow \frac{1}{A_{k,k}} A_{k+1:n,k};$$

$$A_{k+1:n,k+1:n} \leftarrow A_{k+1:n,k+1:n} - A_{k+1:n,k} A_{k,k+1:n};$$

}

Answer the following questions.

(1) Compute the LU factorization of the following matrix.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

(2) Prove that $L_{i,j} = 0$ and $U_{i,j} = 0$ if $|i - j| > w$.

(3) Assume that $n \gg 1$, $w \ll n$, and $m \gg n$. We wish to solve a set of linear systems,

$$A\vec{x}_i = \vec{b}_i, \quad i = 1, 2, \ldots, m,$$

where $\vec{b}_i$ $(i = 1, 2, \ldots, m)$ are the constant vectors, and $\vec{x}_i$ $(i = 1, 2, \ldots, m)$ are the unknown vectors. Explain the relative merits of the following methods (I) and (II) with respect to the amount of arithmetic operations.

(I) Compute A^{-1} first, and then compute each $\vec{x}_i$ as $A^{-1}\vec{b}_i$.

(II) Compute the LU factorization of A first, and then solve $L\vec{y}_i = \vec{b}_i$ for $\vec{y}_i$. Solve $U\vec{x}_i = \vec{y}_i$ for $\vec{x}_i$ lastly.

Problem 2

Let $G = (V, E)$ be a simple undirected graph with the vertex set $V = \{v_i \mid i = 1, \ldots, n\}$ and edge set E. For an n-dimensional vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \{-1, +1\}^n$, define $f_G(\boldsymbol{x})$ by

$$f_G(\boldsymbol{x}) = \sum_{(v_i, v_j) \in E} \frac{1 - x_i x_j}{2}.$$

Answer the following questions.

(1) For the case where G is a complete graph K_n of n vertices, compute $a_n = \max_{\boldsymbol{x} \in \{-1,+1\}^n} f_G(\boldsymbol{x})$.

(2) Let K_n and a_n be those given in question (1). Let b_n be the number of edges of K_n. Compute $\lim_{n \to \infty} \frac{a_n}{b_n}$.

(3) Let G be an arbitrary simple undirected graph. When each x_i takes a value of either -1 or $+1$ with probability $\frac{1}{2}$ independently, compute the expected value of $f_G(\boldsymbol{x})$. You may use the linearity of expectation.

(4) Show that, for any simple undirected graph G, there exists some $\boldsymbol{x} \in \{-1, +1\}^n$ such that $f_G(\boldsymbol{x}) \geq \frac{|E|}{2}$. Here, $|E|$ denotes the number of edges of G.

Problem 3

Let $\Sigma = \{a, b\}$. Answer the following questions.

(1) Give a non-deterministic finite state automaton with 3 states that accepts the following language.

$$\{wba \mid w \in \Sigma^*\}$$

(2) Show the minimal deterministic finite state automaton that accepts the language given in question (1).

(3) Prove that the following language L over Σ is not regular. You may use the pumping lemma for regular languages.

$$L = \{w^R baw \mid w \in \Sigma^*\}$$

Here, w^R denotes the reverse of w. For example, $(abb)^R = bba$.

(4) Is the language L in question (3) a context-free language? If it is, construct a context-free grammar that generates L. If not, prove that L is not a context-free language.

Problem 4

Answer the following questions on digital circuits.

(1) Provide a Boolean expression of the output D according to the following truth table. Design and depict a corresponding combinational circuit by using at most six 2-input NAND gates.

Truth table

Input			Output
A	B	C	D
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(2) Depict the internal structure of a D-flip-flop, and explain how the D-flip-flop holds a 1-bit value.

(3) Consider a clock-synchronous sequential circuit with a 1-bit input CLK, a 1-bit input X, and a 1-bit output Y, where the input CLK is used for the clocking. The output Y is '1' when the number of '1' in the input X values in the past three clock cycles (excluding the current clock cycle) is greater than the number of '0'. Otherwise, the output Y is '0'. The output Y may be any value during the initial three clock cycles after the circuit is powered on. Assume that the circuit satisfies the setup-time and hold-time constraints. Design and depict the circuit. You may use at most two D-flip-flops and an arbitrary number of 2-input AND gates, 2-input OR gates, and NOT gates, if necessary.