Problem 1

Ordered binary trees are trees in which each node has at most two ordered children. Below, $\mathcal{T}_n$ denotes the set of all the ordered binary trees with n leaves. Let $d_T(v)$ denote the depth of node v in an ordered binary tree T, *i.e.*, the number of edges on the path from the root to v.

For a sequence $\mathbf{P} = (c_1, c_2, \ldots, c_n)$ of n positive real numbers, we define $S_\mathbf{P}$ and $H_\mathbf{P}$ by:

$$S_\mathbf{P} = \sum_{i=1}^{n} c_i, \quad H_\mathbf{P} = -\sum_{i=1}^{n}(c_i \cdot \log_2(c_i/S_\mathbf{P})).$$

For an ordered binary tree $T \in \mathcal{T}_n$, we define $W_\mathbf{P}(T)$ by:

$$W_\mathbf{P}(T) = \sum_{i=1}^{n}(c_i \cdot d_T(v_i)),$$

where v_i is the i-th leaf from left in T.

Answer the following questions.

(1) Give the tree $T \in \mathcal{T}_4$ that has the smallest value of $W_\mathbf{P}(T)$ in case $\mathbf{P} = (4, 2, 1, 1)$.

(2) Show that $\sum_{i=1}^{n} 2^{-d_T(v_i)} \leq 1$ holds for any ordered binary tree $T \in \mathcal{T}_n$ with leaves $v_1, v_2, \ldots, v_n$.

(3) Assume that $x_1, x_2, \ldots, x_n$ range over the set of positive real numbers so that $\sum_{i=1}^{n} x_i = 1$. Show that $\sum_{i=1}^{n}(c_i \cdot \log_2 x_i)$ is maximized when $x_i = c_i/S_\mathbf{P}$ for any sequence $\mathbf{P} = (c_1, c_2, \ldots, c_n)$ of n positive real numbers.

(4) Show that any ordered binary tree $T \in \mathcal{T}_n$ satisfies $W_\mathbf{P}(T) \geq H_\mathbf{P}$ for any sequence $\mathbf{P} = (c_1, c_2, \ldots, c_n)$ of n positive real numbers.

Problem 2

Answer the following questions on digital circuits.

(1) Design and depict a circuit equivalent to XOR (exclusive OR) gate by using at most five 2-input NAND gates.

(2) Design and depict a 1-bit full-adder by using only two 2-input XOR gates and three 2-input NAND gates.

(3) Design and depict a 4-bit adder circuit by using four 1-bit full-adders. You may use 2-input NAND gates, 2-input NOR gates, and NOT gates, if necessary. Indicate also the critical path of the 4-bit adder circuit.

(4) Consider a 4-bit clock-synchronous up-down binary counter circuit. The circuit has a 1-bit input CLK for the clocking. The circuit also has a 1-bit input X and a 4-bit output Y. The circuit counts a number from 0 to 15, and outputs the counter value to the output Y. When the input X is '1', the counter value is incremented by one for each positive clock edge. Otherwise, the counter value is decremented by one for each positive clock edge. The circuit allows overflows, *i.e.* the next counter value is 0 when the current counter value is 15 and the input X is '1', and the next counter value is 15 when the current counter value is 0 and the input X is '0'. Assume that the circuit satisfies the setup-time and hold-time constraints. Design and depict the 4-bit clock-synchronous up-down binary counter circuit. You may use 1-bit full-adders, D-flip-flops, 2-input NAND gates, 2-input NOR gates, and NOT gates, if necessary.

Problem 3

Let Σ be the set $\{a, b\}$ of letters. Given two languages $L_1, L_2 \subseteq \Sigma^*$, we define $L_1 \lhd L_2$ by:

$$L_1 \lhd L_2 = \{w \in \Sigma^* \mid \exists v \in L_1.vw \in L_2\}.$$

For example, if $L_1 = \{ab, bb\}$ and $L_2 = \{aa, abb, bbab\}$, then

$$L_1 \lhd L_2 = \{b, ab\}.$$

For a finite automaton $\mathcal{A}$, we write $\mathcal{L}(\mathcal{A})$ for the language accepted by $\mathcal{A}$.
Answer the following questions.

(1) Let $L_3 = \{aa, b, bb\}$ and $L_4 = \{a, b, ab, bb, aaa, bbab\}$. Give the set $L_3 \lhd L_4$.

(2) Let L_5 and L_6 be the languages expressed by the regular expressions $(a^*b)^*$ and $(abba)^*$, respectively. Express $L_5 \lhd L_6$ by using a regular expression.

(3) Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_{1,0}, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_{2,0}, F_2)$ be deterministic finite automata. Here, $Q_i, \delta_i, q_{i,0}$, and F_i are the set of states, the transition function, the initial state, and the set of final states of $\mathcal{A}_i$ ($i \in \{1, 2\}$), respectively. Assume that the transition functions $\delta_i \in Q_i \times \Sigma \to Q_i$ ($i \in \{1, 2\}$) are total. Give a non-deterministic finite automaton that accepts $\mathcal{L}(\mathcal{A}_1) \lhd \mathcal{L}(\mathcal{A}_2)$, with a brief explanation. You may use ϵ-transitions.

(4) Answer whether the following statement is true:

> "For every context-free language L and regular language R, $L \lhd R$ is a regular language."

Also, give a proof sketch if the answer is yes, and give a counterexample if the answer is no.

Problem 4

Let n and d ($n < d$) be natural numbers and $\mathbb{R}$ be the set of real numbers. Denote by $\top$ the transposition operator of a vector and a matrix. Define the inner product of two column vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^d$ as $\boldsymbol{x}_1^\top \boldsymbol{x}_2 \in \mathbb{R}$. Let $\boldsymbol{w} = (w_1, w_2, \ldots, w_d)^\top \in \mathbb{R}^d$ be a d-dimensional column vector, $X \in \mathbb{R}^{n \times d}$ an $n \times d$ matrix where $XX^\top$ is invertible, and $\boldsymbol{y} \in \mathbb{R}^n$ an n-dimensional column vector. Consider solving the following optimization problem by using the Lagrange multipliers method.

$$\min_{\boldsymbol{w}} \frac{1}{2}\|\boldsymbol{w}\|^2 \text{ subject to } \boldsymbol{y} = X\boldsymbol{w},$$

where $\|\boldsymbol{w}\| = \sqrt{w_1^2 + w_2^2 + \ldots + w_d^2}$. The Lagrange function is given by

$$L(\boldsymbol{w}, \boldsymbol{\mu}) = \frac{1}{2}\|\boldsymbol{w}\|^2 + \boldsymbol{\mu}^\top(\boldsymbol{y} - X\boldsymbol{w}),$$

where $\boldsymbol{\mu} \in \mathbb{R}^n$ is the Lagrange multipliers.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive real values. The sets of column vectors $\{\boldsymbol{u}_i \in \mathbb{R}^n\}_{i=1}^n$ and $\{\boldsymbol{v}_j \in \mathbb{R}^d\}_{j=1}^d$ form an orthonormal basis of $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively; that is, they are all unit vectors and orthogonal to each other. Suppose that the singular value decomposition of X is

$$X = U\Lambda V^\top,$$

where U is an $n \times n$ matrix, Λ is an $n \times d$ matrix, and V is a $d \times d$ matrix given by

$$U = (\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots \boldsymbol{u}_n), \ \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & \vdots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n & 0 & \cdots & 0 \end{pmatrix}, \ V^\top = \begin{pmatrix} \boldsymbol{v}_1^\top \\ \boldsymbol{v}_2^\top \\ \vdots \\ \boldsymbol{v}_d^\top \end{pmatrix}.$$

Moreover, define

$$X^- = V(\Lambda^-)^\top U^\top, \text{ where } \Lambda^- = \begin{pmatrix} 1/\lambda_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & 0 & \cdots & 0 & \vdots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 1/\lambda_n & 0 & \cdots & 0 \end{pmatrix}.$$

Answer the following questions. Describe not only an answer but also the derivation process.

(1) Express XX^-X using only X.

(2) Express $XX^\top$ using only U and λ_i ($i = 1, \ldots, n$).

(3) Suppose we wish to express the stationary points of $L(\boldsymbol{w}, \boldsymbol{\mu})$ in the form of $\boldsymbol{w} = A\boldsymbol{y}$ and $\boldsymbol{\mu} = B\boldsymbol{y}$. Express the matrices $A \in \mathbb{R}^{d \times n}$ and $B \in \mathbb{R}^{n \times n}$ using only X.

(4) Express A in question (3) using only X^-.