(A problem correction was made at the time of examination.)
(An English correction was also made for the clarification.)

2019 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may use the back of the sheet if necessary.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. Do not separate the draft papers from this problem booklet.
7. Any answer sheet including marks, symbols and/or words unrelated to your answer will be invalid.
8. Do not take either the answer sheets or the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

(draft paper)

Problem 1

A complex square matrix X is unitary if it holds that $XX^* = I$, where X^* is the conjugate transpose of X (also known as the adjoint matrix of X) and I is the appropriate identity matrix. Let i be the imaginary unit. Answer the following questions.

(1) For a positive integer n, suppose that A and B are unitary matrices of size n. Show that the matrix AB is also unitary.

(2) For a positive integer n, suppose that C and D are real square matrices of size n. Let F be defined as $F = C + \mathrm{i}D$ and G be defined as

$$G = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}.$$

Show that the matrix G is orthogonal if and only if the matrix F is unitary.

(3) Find the eigenvalues of the following matrix.

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix}$$

(4) For a positive integer n, suppose that the (j,k)-th element q_{jk} of a square matrix Q of size n is given by

$$q_{jk} = \frac{1}{\sqrt{n}} \exp\left(\frac{2\pi\mathrm{i}(j-1)(k-1)}{n}\right).$$

Show that the matrix Q is unitary.

(5) Show that a unitary matrix of size 2 with determinant 1 has a form of

$$H = \begin{pmatrix} \exp(\mathrm{i}\psi)\cos\theta & \exp(\mathrm{i}\psi)\sin\theta \\ -\exp(-\mathrm{i}\psi)\sin\theta & \exp(-\mathrm{i}\psi)\cos\theta \end{pmatrix},$$

where θ and ψ are real numbers.

(6) Find the general form of the unitary matrices of size 2.

Problem 2

The real-valued function $u(x,t)$ is defined for $-\infty < x < \infty$ and $t \geq 0$ with independent variables x and t. Consider to solve the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

under the intial conditions

$$u(x,0) = \exp(-ax^2) \tag{2.2}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \tag{2.3}$$

where a and c are positive real numbers. The imaginary unit is represented by i. Answer the following questions.

(1) Calculate the following formula by using complex integration

$$\int_{-\infty}^{\infty} \exp\left(-a(x+\mathrm{i}d)^2\right) \mathrm{d}x,$$

where d is a real number. The following equation may be used.

$$\int_{-\infty}^{\infty} \exp(-x^2)\mathrm{d}x = \sqrt{\pi}$$

(2) The Fourier transform of $u(x,t)$ with respect to x, $U(k,t)$, is defined as

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) \exp(-\mathrm{i}kx)\mathrm{d}x.$$

You may assume that integration with respect to x and differentiation with respect to t are interchangeable. Also, you may assume that $u(x,t)$ and $\frac{\partial u}{\partial x}(x,t)$ converge to 0 as $x \to \pm\infty$ for an arbitrary t.

(i) Express the partial differential equation of $U(k,t)$ when $u(x,t)$ satisfies Eq. (2.1).

(ii) Show that the solution of (i) takes the following form under the initial condition of Eq. (2.3) using a function $F(k)$ of variable k.

$$U(k,t) = F(k)\cos(kct)$$

(iii) Furthermore, using the initial condition of Eq. (2.2), determine $U(k,t)$ by finding $F(k)$. The result of question (1) may be used.

(3) Find $u(x,t)$ by calculating the inverse Fourier transform of $U(k,t)$ obtained in question (2). The inverse Fourier transform is defined as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,t) \exp(\mathrm{i}kx)\mathrm{d}k.$$

Problem 3

Consider a triangle ABC on a plane as shown in the following figure, where the coordinates of the three vertices are A $(1,0)$, B $(0,1)$ and C $(-1,-1)$, respectively. Let ℓ be a randomly chosen half line starting from the origin $(0,0)$, that is,

$$\ell = \{(r\cos\Theta, r\sin\Theta) \mid r \geq 0\},$$

where Θ is a uniformly distributed random variable on the interval $[0, 2\pi)$. Let Q be the point of intersection of the half line ℓ and the edges of the triangle ABC. Let (X, Y) be the coordinates of Q, where X and Y are random variables. Answer the following questions.

(1) Find the probability of the event that the point Q is located on the segment AB.

(2) Show that the expectation of X given the condition that Q is located on AB is $1/2$, where the fact that the triangle ABC is symmetric with respect to the line $y = x$ may be used.

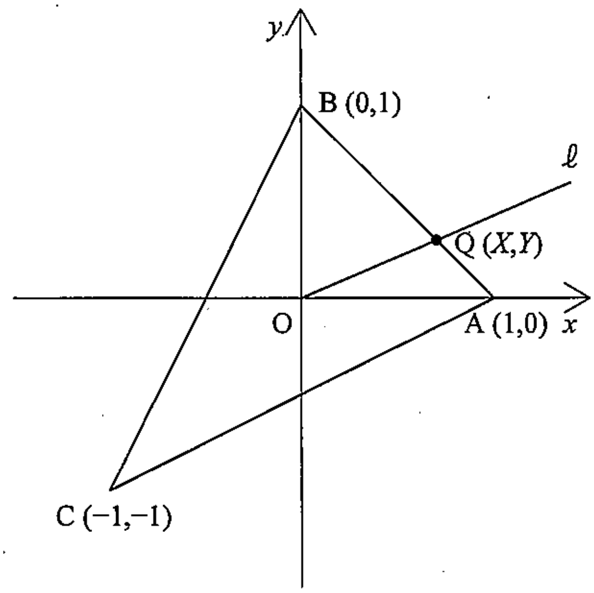
(3) Find the probability density function of X given the condition that Q is located on BC, by using the change-of-variables formula

$$f(x) = g\left(h(x)\right)\left|\frac{\mathrm{d}h}{\mathrm{d}x}(x)\right|$$

where x is an arbitrary real number, f and g are the probability density functions of X and Θ, respectively, and h is the function that satisfies $\Theta = h(X)$.

(4) Let α be the expectation of X given the condition that Q is located on BC. Calculate α by using the result of question (3).

(5) Obtain the expectation μ of X.



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