2017 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

This English translation is supplemental and provided for convenience of applicants. The Japanese version is the official one.

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may use the back of the sheet if necessary.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. Do not separate the draft papers from this problem booklet.
7. Any answer sheet including marks, symbols and/or words unrelated to your answer will be invalid.
8. Do not take either the answer sheets or the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

(draft paper)

Problem 1

Suppose that three-dimensional vectors $\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$ satisfy the equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \qquad (n = 0, 1, 2, \dots),$$

where x_0, y_0, z_0 and α are real numbers, and

$$A = \begin{pmatrix} 1-2\alpha & \alpha & \alpha \\ \alpha & 1-\alpha & 0 \\ \alpha & 0 & 1-\alpha \end{pmatrix}, \qquad 0 < \alpha < \frac{1}{3}.$$

Answer the following questions.

(1) Express $x_n + y_n + z_n$ using x_0, y_0 and z_0.

(2) Obtain the eigenvalues λ_1, λ_2 and λ_3, and their corresponding eigenvectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$ of the matrix A.

(3) Express the matrix A using $\lambda_1, \lambda_2, \lambda_3, \boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$.

(4) Express $\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$ using x_0, y_0, z_0 and α.

(5) Obtain $\lim_{n \to \infty} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$.

(6) Regard

$$f(x_0, y_0, z_0) = \frac{(x_n, y_n, z_n) \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix}}{(x_n, y_n, z_n) \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}}$$

as a function of x_0, y_0 and z_0. Obtain the maximum and the minimum values of $f(x_0, y_0, z_0)$, where we assume that $x_0^2 + y_0^2 + z_0^2 \neq 0$.

Problem 2

A real-valued function $u(x,t)$ is defined in $0 \leq x \leq 1$ and $t \geq 0$. Here, x and t are independent. Suppose solving the following partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad (*)$$

under the following conditions:

$$\text{Boundary condition}: \quad u(0,t) = u(1,t) = 0,$$
$$\text{Initial condition}: \quad u(x,0) = x - x^2.$$

Since the constant function $u(x,t) = 0$ is obviously a solution of the partial differential equation, consider the other solutions. Answer the following questions.

(1) Calculate the following expression, where n and m are positive integers.

$$\int_0^1 \sin(n\pi x)\ \sin(m\pi x)\ \mathrm{d}x$$

(2) Suppose $u(x,t) = \xi(x)\,\tau(t)$, where $\xi(x)$ is a function only of x and $\tau(t)$ is a function only of t. Express the ordinary differential equations for ξ and τ using an arbitrary constant C. You may use that $f(x)$ and $g(t)$ are constant functions when $f(x)$ and $g(t)$ satisfy $f(x) = g(t)$ for arbitrary x and t.

(3) Solve the ordinary differential equations in question (2). Next, show that a solution of partial differential equation $(*)$ which satisfies the boundary condition is given by the following $u_n(x,t)$, and express α and β using a positive integer n.

$$u_n(x,t) = e^{\alpha t} \sin(\beta x)$$

(4) The solution of partial differential equation $(*)$ which satisfies the boundary and initial conditions is represented by the linear combination of $u_n(x,t)$ as shown below. Obtain c_n. You may use the result of question (1).

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

Problem 3

(1) If the probability density function $f(t)$ of a continuous random variable T is denoted by

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

with a positive constant λ, then we say that T follows an exponential distribution with parameter λ. Compute the average of this random variable. Also, derive the probability distribution function $F(t) = P(T \leq t)$ of this exponential distribution, where $P(X)$ is the probability of the event X.

(2) Show that the probability distribution given in question (1) is memoryless. Namely, show that

$$P(T > s + t \mid T > s) = P(T > t)$$

holds for any $s > 0$ and $t > 0$, where $P(X|Y)$ is the probability of the event X conditioned on the event Y.

(3) Let us call the time interval between the time when one starts solving a problem and the time when one finishes it "time required for solution". Assume that the time required for solution of each of n students follows the exponential distribution with the same parameter λ_0. Let all the n students start solving the problem at the same time. Find the probability distribution function and the average of the time required for solution of the student who finishes solving the problem earliest. Here, the time required for solution of each student is mutually independent.

(4) Assume that the times required for solution of a student A and a student B follow the exponential distributions with parameters λ_A and λ_B, respectively. Let the two students start solving the problem at the same time. Find the probability that student A finishes solving the problem earlier than student B.

(5) Let a smart student Hideo and other ten students start solving a problem at the same time. Assume that the time required for solution of each of all the students follows an exponential distribution, where the average time required for solution of each of all the students except Hideo is ten times longer than that of Hideo. Find the probabilities that Hideo finishes solving the problem first and fourth, respectively.

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