2015 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: **10:00 to 12:30**

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take either the answer sheets or the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

Let A and b be defined as

$$A = \begin{pmatrix} -3 & 0 & 0 \\ -2 & -3 & 1 \\ 2 & -3 & -3 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The partial derivative of a scalar-valued function $f(x)$ with respect to $x = (x_1 \; x_2 \; x_3)^T$ is defined as

$$\frac{\partial}{\partial x} f(x) = \left(\frac{\partial}{\partial x_1} f(x) \;\; \frac{\partial}{\partial x_2} f(x) \;\; \frac{\partial}{\partial x_3} f(x) \right),$$

and a stationary point of $f(x)$ is defined as x satisfying $\frac{\partial}{\partial x} f(x) = (0 \; 0 \; 0)$. x^T denotes the transpose of x. Answer the following questions.

(1) Find the characteristic polynomial of A.

(2) C is given as $C = A^5 + 9A^4 + 30A^3 + 36A^2 + 30A + 9I$ by using A and an identity matrix I. Calculate C.

(3) Calculate the partial derivative of $x^T A x$ with respect to x.

(4) Find a symmetric matrix $\tilde{A}$ that satisfies equation $x^T A x = x^T \tilde{A} x$ for any vector x. Find eigenvalues $\lambda_1, \lambda_2, \lambda_3$ ($\lambda_1 \geq \lambda_2 \geq \lambda_3$), and eigenvectors v_1, v_2, v_3. Choose the eigenvectors such that $V = (v_1 \; v_2 \; v_3)$ becomes an orthogonal matrix.

(5) Prove that $x^T A x \leq 0$ holds for any real vector x.

(6) Find a stationary point of function $g(x) = x^T A x + 2b^T x$.

Problem 2

Answer the following questions regarding curves on the xy-plane.

(1) Show that the foci of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ \ (a > b > 0) \qquad (*)$$

and those of a hyperbola:

$$\frac{x^2}{c^2} - \frac{y^2}{d^2} = 1 \ \ (c > d > 0) \qquad (**)$$

are $\left(\pm\sqrt{a^2 - b^2}, 0\right)$ and $\left(\pm\sqrt{c^2 + d^2}, 0\right)$, respectively. Note that an ellipse (hyperbola) is a curve such that the sum (difference) of the distances from the foci to any point on the curve is constant.

(2) As for Eq. $(*)$, consider the set E_u of ellipses such that $a^2 - b^2 = u^2$ (u is a positive constant). By writing the simultaneous equations that consist of Eq. $(*)$ and the differential equation obtained by taking the derivative of Eq. $(*)$ with respect to x, show that any ellipse in E_u satisfies

$$xyy'^2 + \left(x^2 - y^2 - u^2\right) y' - xy = 0, \qquad (***)$$

where $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(3) As for Eq. $(**)$, consider the set H_u of hyperbolae such that $c^2 + d^2 = u^2$. Show that any hyperbola in H_u satisfies Eq. $(***)$.

(4) Let C_u be the set of curves perpendicular to any ellipse in E_u. Let D_u be the set of curves obtained by removing from C_u the line $x = 0$ as well as all the curves including a point such that $y' = 0$. Find a differential equation that any curve in D_u satisfies.

(5) Solve the differential equation that you found in Question (4). If necessary, rewrite the differential equation into a differential equation with respect to p with replacement such that $\alpha = x^2$, $\beta = y^2$, and $p = \dfrac{\mathrm{d}\beta}{\mathrm{d}\alpha}$.

Problem 3

Answer the following questions.

(1) Let X be a real-valued random variable. Let t be a real-valued variable. We define $\phi_X(t)$ for X as

$$\phi_X(t) = E_X[e^{tX}],$$

where $E_X[\cdot]$ denotes the expectation taken with respect to X. Supposing that $\phi_X(t)$ is finite in a neighborhood of $t = 0$, give the mean and variance of X using $\phi'_X(0)$ and $\phi''_X(0)$. Here $\phi'_X(t)$ and $\phi''_X(t)$ denote the first- and second-order derivatives of $\phi_X(t)$ with respect to t, respectively.

(2) For a sequence of mutually independent random variables: $X_1, X_2, \ldots, X_N$, suppose that each X_j is identically generated according to the 1-dimensional normal distribution with mean μ and variance σ^2. That is, the probability density function for each X_j is given by

$$p(X_j = x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Then calculate $\phi_{X_j}(t)$. Also find a probability distribution according to which

$$Y = X_1 + X_2 + \cdots + X_N$$

is generated. You can use the fact that for random variables Z and W with $\phi_Z(t) = \phi_W(t)$, the probability distribution of Z is the same as that of W.

(3) Suppose that $N \in \{1, 2, \ldots, \infty\}$ as in Question (2) is generated according to the geometric distribution with parameter θ $(0 < \theta < 1)$ for which the probability function is given by

$$P(N = n) = (1-\theta)^{n-1}\theta.$$

For $Y = X_1 + X_2 + \cdots + X_N$, define $\phi_Y(t)$ by

$$\phi_Y(t) = E_Y[e^{tY}].$$

Then calculate $\phi_Y(t)$ and express it using $\phi_{X_j}(t)$. Since $\phi_{X_j}(t)$ does not depend on j, you can write it as $\phi_X(t)$.

(4) Calculate the mean and variance of Y in Question (3).

(5) For given $\xi(> E_Y[Y])$, give an upper bound on the probability that Y in Question (3) exceeds ξ, as a function of μ, σ, θ, and ξ (not all of μ, σ, θ, and ξ have to be used).

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