2014 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take either the answer sheets or the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

A real square matrix M is said to be symmetric if it satisfies the condition $M = M^T$, where M^T denotes the transpose of M. Answer the following questions.

(1) Find all the eigenvalues and their corresponding eigenvectors of the symmetric matrix A given as follows.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

(2) Prove that if a real square matrix M is symmetric, then all of its eigenvalues are real.

(3) Even if all the eigenvalues of a real square matrix M are real, M is not necessarily symmetric. Provide a concrete example of such a matrix M.

(4) Let $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a non-zero three dimensional real vector. For the symmetric matrix A defined in question (1), let us define the function $f(x, y, z)$ as follows.

$$f(x, y, z) = \frac{\boldsymbol{u}^T A \boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{u}}$$

Here, $\boldsymbol{u}^T$ denotes the transpose of $\boldsymbol{u}$. Moreover, the function $g(x, y, z)$ is defined as follows.

$$g(x, y, z) = \frac{xy + yz}{x^2 + y^2 + z^2}$$

Show that the following equation holds.

$$f(x, y, z) = 1 - 2g(x, y, z)$$

(5) Using eigenvalue decomposition of the symmetric matirx A defined in question (1), show that the following inequality holds for the function $g(x, y, z)$ defined in question (4).

$$-\frac{1}{\sqrt{2}} \leq g(x, y, z) \leq \frac{1}{\sqrt{2}}$$

Problem 2

For real functions $f(x)$ and $g(x)$ defined on $[-1, 1]$, let

$$(f, g) = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

Answer the following questions.

(1) Calculate (q_0, q_1), (q_0, q_2), (q_1, q_2), (q_0, q_0), (q_1, q_1), and (q_2, q_2) for polynomials defined by

$$q_0(x) = 1, \quad q_1(x) = x, \quad q_2(x) = 3x^2 - 1.$$

You may use the fact that $\int_{-1}^{1} h(x)\mathrm{d}x = 0$ holds for any odd function $h(x)$.

(2) Let $p_k(x)$ be a polynomial of degree k with respect to x (with the coefficient of x^k not being 0), and consider a series of polynomial functions $p_0(x), p_1(x), \ldots$. In what follows, we say a set of N functions $\{p_0(x), p_1(x), \ldots, p_{N-1}(x)\}$ satisfies the orthonormal condition when

$$(p_i, p_j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

holds for any integers i and j in $[0, N-1]$.

(2-1) Using $q_0(x), q_1(x)$, and $q_2(x)$ defined in question (1), find a set of functions $\{p_0(x), p_1(x), p_2(x)\}$ satisfying the orthonormal condition.

(2-2) Find a function $p_3(x)$ so that a set of functions $\{p_0(x), p_1(x), p_2(x), p_3(x)\}$ satisfies the orthonormal condition, where $p_0(x), p_1(x)$, and $p_2(x)$ are the functions derived in question (2-1).

(3) Suppose that a set of functions $\{p_0(x), p_1(x), \ldots, p_{N-1}(x)\}$ satisfies the orthonormal condition. Then, in a manner similar to question (2-2), a function $p_N(x)$ can be found such that the set of functions $\{p_0(x), p_1(x), \ldots, p_N(x)\}$ satisfies the orthonormal condition. Show that $p_N(x)$ is unique except the sign, following the steps below.

(3-1) Generally, a polynomial $f_N(x)$ of degree N can be written as

$$f_N(x) = \sum_{k=0}^{N} c_k p_k(x).$$

Express the coefficients c_k $(k = 0, \ldots, N)$ with $p_0(x), p_1(x), \ldots, p_N(x)$, and $f_N(x)$.

(3-2) Suppose that there exists a polynomial function $\tilde{p}_N(x)$ of degree N different from $p_N(x)$ such that the set of functions $\{p_0(x), p_1(x), \ldots, p_{N-1}(x), \tilde{p}_N(x)\}$ also satisfies the orthonormal condition. Then prove that $\tilde{p}_N(x) = -p_N(x)$, by considering the case $f_N(x) = \tilde{p}_N(x)$ in question (3-1).

Problem 3

Let $x_0, x_1, x_2, \ldots$ be a sequence of independent random variables. Each x_i $(i = 0, 1, 2, \ldots)$ takes value 1 with probability p and value 0 with probability $1 - p$. Answer the following questions.

(1) With regards to the sequence of random variables $x_0, x_1, x_2, \ldots$, answer the following questions.

(1-1) Calculate the variance of x_i, and obtain the probability that $x_0 = x_1 = 1$.

(1-2) Let k $(k \geq 0)$ be the smallest integer such that $x_k = x_{k+1}$. For example, if $x_0, x_1, x_2, \ldots$ is $1, 0, 0, 1, 0, 1, 1, 1, 0, 0, \ldots$, then $k = 1$ and $x_k = 0$.

Obtain the probability that $x_k = 1$.

(2) A sequence of random variables $y_0, y_1, y_2, \ldots$ is defined using $x_0, x_1, x_2, \ldots$ as follows.

$$\begin{aligned} y_0 &= 1 \\ y_{i+1} &= y_i + \alpha(x_i - y_i) \quad (i = 0, 1, 2, \ldots) \end{aligned}$$

Assume $0 < \alpha < 1$, and answer the following questions.

(2-1) Show that $y_n = (1 - \alpha)^n + \sum_{i=0}^{n-1} (1 - \alpha)^{n-i-1} \alpha x_i$ $(n = 1, 2, \ldots)$.

(2-2) Obtain the expected value E_n and the variance V_n of y_n.

(2-3) Let $E_\infty = \lim_{n\to\infty} E_n$ and $V_\infty = \lim_{n\to\infty} V_n$. Assuming $\frac{1}{2} < p < \frac{3}{4}$, obtain the maximum value of α that satisfies the condition

$$E_\infty - \sqrt{V_\infty} \geq \frac{1}{2}.$$

(3) A sequence of random variables $z_0, z_1, z_2, \ldots$ is defined using $x_0, x_1, x_2, \ldots$ as follows. If x_j, x_{j+1} is the i-th pair of adjacent variables in $x_0, x_1, x_2, \ldots$ such that $x_j = x_{j+1}$, then z_i is defined by $z_i = x_j$. The index i starts with 0, so if $x_j = x_{j+1}$ is the first such pair, then $z_0 = x_j$. For example, if $x_0, x_1, x_2, \ldots$ is $1, 0, 0, 1, 0, 1, 1, 1, 0, 0, \ldots$, then $z_0 = 0, z_1 = 1, z_2 = 1, z_3 = 0, \ldots$.

Let q_i be the probability that $z_i = 1$. Obtain $\lim_{i\to\infty} q_i$.

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