2013 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take either the answer sheets or the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

Answer the following questions.

(1) For $(c_1\ \ c_2\ \ c_3\ \ c_4\ \ c_5\ \ c_6\ \ c_7) = (1\ \ 2\ \ -8\ \ 20\ \ -44\ \ 92\ \ -188)$, there exists a pair of integer k (≥ 1) and $k \times k$ constant matrix $\boldsymbol{M}$ satisfying

$$\begin{pmatrix} c_{i+1} \\ c_{i+2} \\ \vdots \\ c_{i+k} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} c_i \\ c_{i+1} \\ \vdots \\ c_{i+k-1} \end{pmatrix}$$

for any $i = 1, 2, \ldots, 7-k$. Among such pairs, find a pair with the smallest k by examining cases $k = 1, 2, \ldots$ in order.

(2) The numbers c_i $(i = 1, 2, \ldots, 7)$ in question (1) can be represented by $c_i = \boldsymbol{L}\boldsymbol{M}^{i-1}\boldsymbol{N}$, where k and $\boldsymbol{M}$ are the pair obtained in question (1), and $\boldsymbol{L}$ $(\in \mathbb{R}^{1\times k})$ and $\boldsymbol{N}$ $(\in \mathbb{R}^{k\times 1})$ are real constant vectors. Find such a pair of $\boldsymbol{L}$ and $\boldsymbol{N}$.

(3) Define $c_8, c_9, \ldots$ by $c_i = \boldsymbol{L}\boldsymbol{M}^{i-1}\boldsymbol{N}$, where $\boldsymbol{M}$, $\boldsymbol{L}$, and $\boldsymbol{N}$ are the ones obtained in questions (1) and (2). Find the rank of the following matrix $\boldsymbol{C}$ together with the derivation:

$$\boldsymbol{C} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \cdots & c_{10} \\ c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & \cdots & c_{11} \\ c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & \cdots & c_{12} \\ c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & \cdots & c_{13} \\ c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & \cdots & c_{14} \\ c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & \cdots & c_{15} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & \cdots & c_{19} \end{pmatrix}$$

(4) Define a function $f(x) = \boldsymbol{L}(\boldsymbol{I} - x\boldsymbol{M})^{-1}\boldsymbol{N}$, where x is a scalar variable, $\boldsymbol{M}$, $\boldsymbol{L}$, and $\boldsymbol{N}$ are the ones obtained in questions (1) and (2), and $\boldsymbol{I}$ is a $k \times k$ identity matrix. Let $f(x) = f_1 + f_2 x + f_3 x^2 + \cdots$ be the Taylor series of $f(x)$ at $x = 0$. Represent f_i $(\in \mathbb{R})$, $i = 1, 2, \ldots$ by using c_i $(= \boldsymbol{L}\boldsymbol{M}^{i-1}\boldsymbol{N})$, $i = 1, 2, \ldots$ together with the derivation. Use the following formula of diagonal matrix $\boldsymbol{D}$ if necessary:

$$\left.\frac{\mathrm{d}^i}{\mathrm{d}x^i}(\boldsymbol{I} - x\boldsymbol{D})^{-1}\right|_{x=0} = i!\boldsymbol{D}^i,\ i = 1, 2, \ldots .$$

Problem 2

Let $\mathcal{F}$ be the set of functions $f = f(x)$ on the real line with $f(0) = 0$ and $f(1) = 1$. For $f \in \mathcal{F}$, we define $I = I[f]$ by

$$I[f] = \int_0^1 \left[\{f(x)\}^2 + \left\{ \frac{\mathrm{d}f(x)}{\mathrm{d}x} \right\}^2 \right] \mathrm{d}x.$$

We are interested in a function in $\mathcal{F}$ that minimizes I. Answer the following questions, where every function considered in this problem is assumed to be sufficiently smooth everywhere.

(1) Show that, for any $f, g \in \mathcal{F}$ and $t \in [0, 1]$,

$$I[(1-t)f + tg] \leq (1-t)I[f] + tI[g]$$

holds.

(2) Consider $f \in \mathcal{F}$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} I[(1-t)f + tg] \Big|_{t=0} = 0$$

for any $g \in \mathcal{F}$. Derive an ordinary differential equation that f should satisfy. If necessary, the following property may be used.

If function F satisfies

$$\int_0^1 G(x)F(x)\mathrm{d}x = 0$$

for any function G with $G(0) = G(1) = 0$, then $F(x) = 0$ for $x \in [0, 1]$.

(3) Explain the reason why the solution of the ordinary differential equation obtained in question (2) minimizes I.

(4) Find the solution of the ordinary differential equation obtained in question (2).

Problem 3

Initially, a bag contains only a black ball. Consider the repetition of the following operation.

Operation A: A ball is randomly drawn from the bag. If the color of the ball is black, put it back with a ball of a new non-black color. Otherwise, put it back with another ball of the same color.

Note that a ball is not distinguishable from the others until it is drawn from the bag. Answer the following questions.

(1) Consider the case where *Operation A* is repeated 3 times. Find the probability that the number of colors of the balls in the bag (black is not counted) is 2 and 3, respectively.

(2) Consider the case where *Operation A* is repeated 4 times. Find the probability that the number of colors of the balls in the bag (black is not counted) is 3.

(3) Consider the case where *Operation A* is repeated n times. Show that $1/n$ is the probability that the number of colors of the balls in the bag (black is not counted) is 1.

(4) Consider the case where *Operation A* is repeated n times. Find the probability $q_n(m)$ that the number of colors of the balls in the bag (black is not counted) is m. If necessary, use $S(n, m)$ defined by the following equation.

$$S(n,m) = \begin{cases} S(n-1, m-1) + (n-1)S(n-1, m), & n > m > 0 \\ 1, & n = m \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(5) Find the minimum number of trials of *Operation A* such that the probability exceeds 35% for the case where the number of colors of the balls in the bag (black is not counted) is at least 3. Explain the derivation.

(6) Show that for all natural numbers n, $q_n(m)$ defined in question (4) satisfies the following equality.

$$\sum_{m=1}^{n} m q_n(m) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

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