2012 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take the answer sheets and the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

A square matrix D is *orthogonal* when $DD^\top = D^\top D = I$. Here $D^\top$ denotes the transpose of D; I is the unit (or identity) matrix. The following fact can be used without a proof: if $D^\top D$ is the unit matrix, so is $DD^\top$.

Let A be the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Answer the following questions.

(1) Find eigenvalues and eigenvectors of the matrix $A^\top A$.

(2) Find real numbers $\lambda_1, \lambda_2, \lambda_3$ and an orthogonal matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$$

such that

$$A^\top A = U \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} U^\top.$$

Choose your answer so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $u_{31} \geq 0, u_{32} \geq 0, u_{33} \geq 0$. Note that $\mathbf{u}_i$ denotes the following 3×1 column vector:

$$\mathbf{u}_i = \begin{pmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \end{pmatrix}.$$

(3) Find a matrix B such that $B^2 = A^\top A$. Choose B so that its eigenvalues are all positive.

(4) Let C be the following 3×3 matrix:

$$C = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}}\mathbf{u}_1 & \frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2 & \frac{1}{\sqrt{\lambda_3}}\mathbf{u}_3 \end{pmatrix}.$$

Show that the matrix AC is orthogonal.

(5) Find orthogonal matrices V and W such that

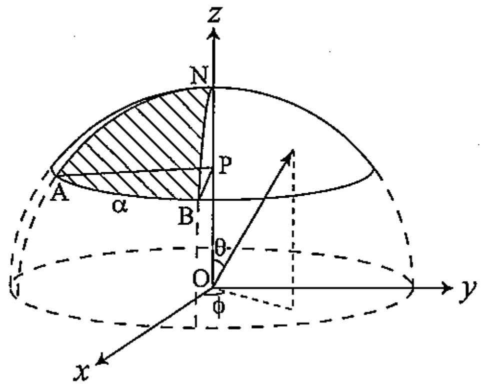
$$A = V \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{pmatrix} W.$$

Problem 2

Let H be the upper hemisphere of the unit sphere centered at the origin in the xyz space. That is,

$$H = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}.$$

Consider the intersection of H and a plane $z = \cos\alpha$, which is a circle, and intercept the circle to yield an arc of length α. Let A and B be the two end points of the arc, as shown in the figure below. Here, α satisfies $0 < \alpha < \frac{\pi}{2}$. Let N $= (0, 0, 1)$ and P $= (0, 0, \cos\alpha)$.



Figure

Answer the following questions.

(1) Find $\angle$APB, the measure of the angle APB.

(2) Let $T(\alpha)$ be the area of the triangle NAB (the plane shape surrounded by the three line segments NA, AB, and BN). Show that

$$\frac{T(\alpha)}{\alpha^2}$$

converges as $\alpha \to 0$ and find its limit value.

(3) Let H_α be a subset of H on and above the plane $z = \cos\alpha$. That is,

$$H_\alpha = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1,\ z \geq \cos\alpha\}.$$

Express coordinates of points on H_α using the two parameters θ and ϕ in the figure. Also show the ranges of values the two parameters can take.

(4) Find the surface area of H_α.

(5) Consider a surface in H surrounded by three arcs, arc AB, arc BN, and arc NA (hatched in the figure), where arc BN (NA) is the curve that connects B and N (N and A) in the shortest distance on H (a part of a great circle). Let $S(\alpha)$ be its surface area. Show that

$$\frac{S(\alpha)}{\alpha^2}$$

converges as $\alpha \to 0$ and find its limit value.

Problem 3

Consider a brand of candy bar. Each bar contains a card chosen out of K different types with equal probability. A prize is offered when r different types of cards ($K \geq r$) are obtained. A customer buys bars one by one, and let $p(n,r)$ be the probability of obtaining cards of r different types *for the first time* after buying n bars. Answer the following questions.

(1) Consider the following description of $p(n,r)$:

$$p(n,r) = \sum_{i=1}^{n+1-r} C_i \; p(n-i, r-1).$$

Express C_i in terms of K, r, and i.

(2) Consider the following description of $p(n,r)$:

$$p(n,r) = A \; p(n-1,r) + B \; p(n-1,r-1).$$

Express A, B in terms of K and r.

(3) Let $P(z,r)$ be the following polynomial expression of z:

$$P(z,r) = \sum_{n=0}^{\infty} p(n,r) z^n.$$

Show that the following equality holds between $P(z,r)$ and $P(z,r-1)$:

$$(K-(r-1)z)P(z,r) = (K-r+1)zP(z,r-1).$$

(4) Show that the expected number of bars that a customer must buy in order to obtain the prize is $P'(1,r)$. Note that P' is the derivative of P with respect to z.

(5) Find the expected number of bars that a customer must buy in order to obtain the prize when $K = r = 7$.

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