# 2011 School Year

# Graduate School
# Entrance Examination Problem Booklet

# Mathematics

Examination Time: 10:00 to 12:30

## Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take the answer sheets and the problem booklet out of the examination room.

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|---|---|
| Examinee's number | No. |

Fill this box with your examinee's number.

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## Problem 1

Define matrix $A$, matrix $B$ and function $f(n)$ as follows:

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & a & b & b \\ 0 & 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & b & a \\ a & b & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 & 0 \\ b & b & a & 0 & 0 & 0 \end{pmatrix}$$

$$f(n) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} B^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $a$ and $b$ are real numbers such that $a > 0$, $b > 0$ and $a \neq b$, and $n$ is a positive integer. Answer the following questions.

(1) Find all eigenvalues of matrix $A$ and show three linearly independent eigenvectors of matrix $A$.

(2) Find all eigenvalues of matrix $B$ and show six linearly independent eigenvectors of matrix $B$.

(3) Find $f(1)$, $f(2)$ and $f(3)$.

(4) Find $f(n)$ when $a = 3$ and $b = 2$.

## Problem 2

Consider the following differential equation defined for $x > 0$:

$$x\frac{d^2y}{dx^2} + (x+4)\frac{dy}{dx} + 3y = 4x + 4.$$

Answer the following questions.

(1) Find the value of $r$ such that $v = x^r$ is a solution of

$$x\frac{d^2v}{dx^2} + (x+4)\frac{dv}{dx} + 3v = 0.$$

(2) For the value of $r$ obtained in question (1), we set $y = x^r u$. Derive the differential equation that $u$ satisfies. Then, show that $u = x^4$ is a solution of the obtained differential equation.

(3) Find the general expression of $\dfrac{du}{dx}$ that satisfies the differential equation obtained in question (2).

(4) Find the general solution $y$.

## Problem 3

Let $B_i$ ($i$: natural number) be a random variable which takes 1 with probability $p$ and takes $-1$ with probability $1-p$ where $0 < p < 1$. Assume that $B_i$ and $B_j$ are independent if $i \neq j$. Let $S_N = \sum_{i=1}^{N} B_i$ ($N$: natural number) and $E[\cdot]$ denote expectation. Answer the following questions.

(1) Find all possible values that $S_4$ can take, together with their probabilities.

(2) Find the conditional probability that $B_1 = 1$ under $S_4 = 2$.

(3) Show that $E[{B_i}^m {B_j}^n] = E[{B_i}^m]E[{B_j}^n]$ for any natural numbers $m$ and $n$ if $i \neq j$.

(4) Find the mean $\mu$ and the variance $\sigma^2$ of $S_N$ when $p = \frac{1}{2}$.

(5) Find $E[{S_2}^4]$ when $p = \frac{1}{2}$.

(6) Find $K_N = \frac{E[{S_N}^4]}{E[{S_N}^2]^2}$ and $\lim_{N\to\infty} K_N$ when $p = \frac{1}{2}$.

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