2010 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take the answer sheets and the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

The following propositions (1) through (4) are concerning an integer $m \geq 2$ and an $m \times m$ real matrix A. For each proposition, answer whether it is true or not for arbitrary m and A, and prove your claim. In case you answer it false, show a counterexample. In (3) and (4) below, $\|\cdot\|$ represents the norm of a vector, which is defined by $\|\boldsymbol{x}\| = \sqrt{x_1^2 + \cdots + x_m^2}$ for $\boldsymbol{x} = {}^t(x_1 \ \cdots \ x_m)$.

(1) If A has m distinct real eigenvalues $\lambda_1, \ldots, \lambda_m$, its eigenvectors are linearly independent. That is, suppose x_i is the eigenvector corresponding to λ_i. Then,

$$\alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_m \boldsymbol{x}_m = \boldsymbol{0} \quad \text{implies} \quad \alpha_1 = \cdots = \alpha_m = 0$$

holds for arbitrary real numbers $\alpha_1, \ldots, \alpha_n$.

(2) If the characteristic polynomial $\det(\lambda I - A)$ of A has a multiple root (a root of multiplicity > 1), A cannot be diagonalized.

(3) If A has m distinct real eigenvalues, the sequence of vectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots$ defined by the following converges for any arbitrary real m-vector $\boldsymbol{b}$.

$$\begin{aligned} \boldsymbol{u}_0 &= \boldsymbol{b}, \\ \boldsymbol{u}_{n+1} &= \begin{cases} A\boldsymbol{u}_n / \|A\boldsymbol{u}_n\| & (\text{if } \|A\boldsymbol{u}_n\| \neq 0) \\ \boldsymbol{0} & (\text{if } \|A\boldsymbol{u}_n\| = 0) \end{cases} \qquad (n = 0, 1, \ldots). \end{aligned}$$

(4) If A has an eigenvalue of multiplicity m and it is a positive real number, the sequence of vectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots$ defined by the following converges for any arbitrary real m-vector $\boldsymbol{b}$.

$$\begin{aligned} \boldsymbol{u}_0 &= \boldsymbol{b}, \\ \boldsymbol{u}_{n+1} &= \begin{cases} A\boldsymbol{u}_n / \|A\boldsymbol{u}_n\| & (\text{if } \|A\boldsymbol{u}_n\| \neq 0) \\ \boldsymbol{0} & (\text{if } \|A\boldsymbol{u}_n\| = 0) \end{cases} \qquad (n = 0, 1, \ldots). \end{aligned}$$

Problem 2

Answer the following questions.

(1) Find the following indefinite integrals:

(i) $\displaystyle\int \frac{1}{x^2+1}dx,$

(ii) $\displaystyle\int \frac{x}{x^2+1}dx.$

(2) Find the following path integrals I_1, I_2, I_3, and I_4, respectively over paths S_1, S_2, S_3 and S_4 in the complex plane shown in Fig. 1.

$$I_k = \int_{S_k} \frac{1}{z-\alpha}dz, \ k = 1,\ 2,\ 3,\ 4,$$

where $\alpha = a + ib$ (a, b: real number, i: imaginary unit), $|a| < 1$ and $|b| < 1$. And also find $I = \sum_{k=1}^{4} I_k$.

(3) Let S denote a contour, i.e., closed path composed of S_k, $k = 1,\ 2,\ 3$ and 4 in Fig. 1. Find the following path integral:

$$\int_S \frac{2z+0.4}{z^2+0.4z+0.05}dz.$$

(4) Assume that a polynomial $f(z)$ of z satisfies

$$f(z) \neq 0 \text{ for any } z \text{ on the contour } S, \text{ and } \int_S \frac{f'(z)}{f(z)}dz = 0,$$

where $f'(z)$ represents the derivative of $f(z)$. Then, show that for the interior region D enclosed by S,

$$f(z) \neq 0,\ \forall z \in D.$$

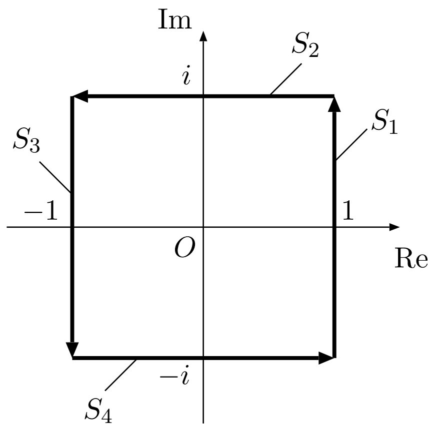

Fig. 1. Paths S_1, S_2, S_3 and S_4 on the complex plane

Problem 3

Answer the following questions.

(1) Let U be a uniformly distributed random variable of the open interval $(0, 1)$. Obtain

(i) the distribution function $F(x) = P(X \leq x)$, and

(ii) the density function $f(x) = \dfrac{d}{dx}F(x)$

of the random variable $X = -\dfrac{1}{\lambda}\log_e U$, where λ is a positive constant value. Here, $P(X \leq x)$ denotes the probability that $X \leq x$ holds for the random variable X.

(2) Let X_i $(i \geq 1)$ be independent random variables each of which follows the distribution of the density function $\lambda e^{-\lambda x}$ for $x \geq 0$, where λ is a positive constant value. Obtain the density functions of the distribution that the following Y_2, Y_3 and Y_n each follows.

(i) $Y_2 = X_1 + X_2$

(ii) $Y_3 = X_1 + X_2 + X_3$

(iii) $Y_n = \displaystyle\sum_{i=1}^{n} X_i$

(3) Consider Y_n of (2). Let $P(Y_n \geq 1)$ denote the probability that $Y_n \geq 1$ holds. Describe $P(Y_n \geq 1) - P(Y_{n-1} \geq 1)$ as a function of λ and n.

(4) From uniformly distributed random numbers, the following method Q generates a random number that follows a certain distribution, where λ is a positive constant value. Obtain the probability when K's value is k, namely $P(K = k)$.

Procedure Q Generate u_i $(i \geq 1)$ successively that are uniformly distributed random numbers of the open interval $(0, 1)$. Compute z_i $(i \geq 1)$ as follows, and let $K = n - 1$ for the n that fulfils $z_n < e^{-\lambda}$ for the first time.

$$
\begin{aligned}
z_1 &= u_1 \\
z_2 &= z_1 u_2 \\
z_3 &= z_2 u_3 \\
&\vdots \\
z_n &= z_{n-1} u_n \\
&\vdots
\end{aligned}
$$

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