2009 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take the answer sheets and the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

(blank page for rough work)

Problem 1

Suppose that three-dimensional column vectors $\boldsymbol{x}_n$ satisfy the recurrence equation:

$$\boldsymbol{x}_n = A\boldsymbol{x}_{n-1} + \boldsymbol{u} \qquad (n = 1, 2, \ldots),$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & a & a^2 \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}.$$

Let a, b, c and d be real constants and assume $a \neq 0$. Answer the following questions.

(1) Express the eigenvalues of A with a.

(2) Express A's eigenvectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ with a. Let $\boldsymbol{p}$ and $\boldsymbol{r}$ correspond to the largest and the smallest eigenvalues, respectively, and set $\|\boldsymbol{p}\| = \|\boldsymbol{r}\| = \dfrac{1}{\sqrt{a^2+1}}$ and $\|\boldsymbol{q}\| = 1$.

(3) Express $\boldsymbol{u}$ and $\boldsymbol{x}_0$ as linear combinations of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$.

(4) Suppose $\boldsymbol{x}_n = \alpha_n\boldsymbol{p} + \beta_n\boldsymbol{q} + \gamma_n\boldsymbol{r}$. Describe α_n, β_n and γ_n using α_{n-1}, β_{n-1} and γ_{n-1}.

(5) Obtain $\boldsymbol{x}_n$.

(6) Suppose that θ_n denotes the angle between $\boldsymbol{x}_n$ and vector $\boldsymbol{s}$, where

$$\boldsymbol{s} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Describe a necessary and sufficient condition on a, b, c and d for $\lim_{n\to\infty} \theta_n = 0$.

Problem 2

Consider a system whose temperature is controlled by a heater.

Let $x(t)$ denote the temperature of the system at time t. While the switch of the heater is off, temperature $x(t)$ follows the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x. \tag{$*$}$$

While the switch of the heater is on, temperature $x(t)$ follows the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - x. \tag{$**$}$$

First, consider the behavior of the system when the switch is kept intact.

(1) Find the solutions of the differential equations $(*)$ and $(**)$ with the initial condition $x(0) = x_0$.

Next, given the thresholds θ_0 and θ_1 for the temperature that satisfy $0 < \theta_0 < \theta_1 < 1$, consider automatically manipulating the switch according to the following rule to keep the temperature between θ_0 and θ_1. Turn on the switch when the temperature is not higher than θ_0 and the switch is off. Turn off the switch when the temperature is not lower than θ_1 and the switch is on. Otherwise, keep the switch intact.

Assume that the temperature x_0 at time 0 is higher than θ_0, and the switch is off at time 0.

(2) Find time t_1 when the switch is turned on for the first time after time 0.

(3) The temperature evolves periodically after time t_1. Find the period, τ.

(4) Consider the time average of the temperature during one period, $\bar{x} = \dfrac{1}{\tau}\displaystyle\int_{t_1}^{t_1+\tau} x(t)\mathrm{d}t$, and the mean of the thresholds, $\bar{\theta} = \dfrac{\theta_0 + \theta_1}{2}$. The difference between these values $\bar{x} - \bar{\theta} = \dfrac{1}{\tau}\displaystyle\int_{t_1}^{t_1+\tau} (x(t) - \bar{\theta})\mathrm{d}t$ can be expressed with a certain function $f(x, \bar{\theta})$ as follows:

$$\bar{x} - \bar{\theta} = \frac{1}{\tau}\int_{\theta_0}^{\theta_1} f(x, \bar{\theta})\mathrm{d}x.$$

Find the function $f(x, \bar{\theta})$.

(5) Suppose that θ_0 and θ_1 are varied so that the mean $\bar{\theta}$ takes a constant value. Prove that if $\bar{\theta} < \dfrac{1}{2}$, then $(\bar{x} - \bar{\theta})\tau$ monotonically decreases as $w = \dfrac{\theta_1 - \theta_0}{2}$ increases.

Problem 3

Answer the following questions concerning random walk of a particle. The particle moves to an adjacent point with the equal probability at one time.

As shown in Fig. 3.1, consider points with integer coordinates in the range of $0 \leq x \leq n$ $(n \geq 2)$ on the x-axis. Random walk terminates when the particle arrives at $x = 0$ or $x = n$. $P_n(k)$ represents the probability that the particle at $x = k$ $(0 \leq k \leq n)$ terminates its random walk by arriving at $x = n$.

Figure 3.1

(1) For $1 \leq k \leq n-1$, describe $P_n(k)$ using $P_n(k-1)$ and $P_n(k+1)$.

(2) Calculate $P_3(2)$ using the facts $P_3(0) = 0$ and $P_3(3) = 1$.

(3) Find $P_n(k)$.

(4) Let $T_n(k)$ represent the average number of steps the particle at $x = k$ $(0 \leq k \leq n)$ makes before it terminates its random walk. Find $T_n(k)$.

As shown in Fig. 3.2, consider points with integer coordinates in the range of $-2 \leq x \leq 4$ on the x-axis and $-2 \leq y \leq 3$ on the y-axis. For example, the adjacent points of $(0, 1)$ are $(0, 2)$ and $(0, 0)$, and those of the origin $(0, 0)$ are $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Random walk terminates when the particle arrives at one of $(x, y) = (4, 0)$, $(0, 3)$, $(-2, 0)$ and $(0, -2)$.

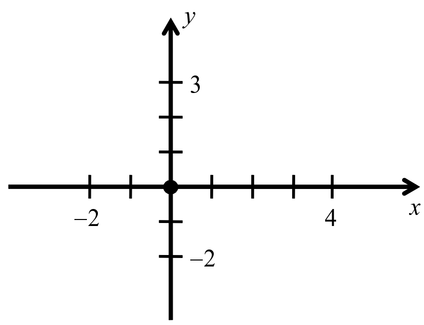

Figure 3.2

(5) Calculate the probability $P_{(4,0)}$ that the particle at the origin $(0, 0)$ terminates its random walk by arriving at $(x, y) = (4, 0)$.

(6) Calculate the average number of steps T that the particle at the origin $(0, 0)$ makes before it terminates its random walk.

(blank page for rough work)

(blank page for rough work)