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Problem 1

Answer the following questions regarding

\[ f(x, y, u, v) = \exp\left( -\frac{(x-u)^2 + (y-v)^2}{2} \right), \]

which is the function on \( \mathbb{R}^4 \). Here, \( \exp p \) denotes \( e^p \) for \( p \in \mathbb{R} \).

(1) Regard \( f(x, y, u, v) \) as the function of \( u, v \). Assume that \( |u| \ll 1 \) and \( |v| \ll 1 \). Then, obtain the Taylor series expansion of \( f(x, y, u, v) \) for \( u, v \) at \( (u, v) = (0, 0) \) up to the first order terms. Hereafter, regard the answer to question (1) as the function of \( (x, y, u, v) \in \mathbb{R}^4 \) and denote it by \( g(x, y, u, v) \).

(2) Let \( (u, v) = (a \cos \theta, a \sin \theta) \), where \( 0 < a \ll 1 \), \( 0 \leq \theta < 2\pi \). Then, obtain

\[ h(x, y) = \int_0^{2\pi} g(x, y, a \cos \theta, a \sin \theta) \cos \theta \, d\theta. \]

(3) Find \( (x, y) \) that satisfies \( x^2 + y^2 + 2x + 2y = 1 \) and maximizes \( h(x, y) \).
Problem 2

Assume that $A$ is a real non-singular square matrix. For any $A$ that satisfies each of the following conditions (1)–(6), answer whether $A^{-1}$ satisfies the same condition. Prove it when it is true. Otherwise provide a counterexample with a proof that it does not satisfy the condition. $A^\top$ represents the transpose of $A$. The $(i, j)$ entry of $A$ is represented as $a_{i,j}$.

1. Normal matrix, that is, $AA^\top = A^\top A$.
2. Lower triangular matrix, that is, $a_{i,j} = 0$ for $i < j$.
3. Unit antidiagonal matrix, that is, $a_{i,n-i+1} = 1$ and other entries are 0, where $n$ is the order of $A$ (i.e., $A$ is an $n \times n$ matrix). An example for $n = 4$ is the following.

$$A = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}$$

4. Positive matrix, that is, $a_{i,j} > 0$ for all $i$ and $j$.
5. Tridiagonal matrix, that is, $a_{i,j} = 0$ for $|i - j| > 1$. An example is the following.

$$A = \begin{pmatrix}
a_{1,1} & a_{1,2} & 0 \\
a_{2,1} & a_{2,2} & a_{2,3} \\
0 & a_{3,2} & a_{3,3}
\end{pmatrix}$$

6. Persymmetric matrix, that is, $a_{i,j} = a_{n-j+1,n-i+1}$ for all $i$ and $j$, where $n$ is the order of $A$. It is symmetric about the antidiagonal (i.e., the diagonal from the top right corner to the bottom left corner). An example for $n = 4$ is the following.

$$A = \begin{pmatrix}
\zeta & \delta & \beta & \alpha \\
\theta & \epsilon & \gamma & \beta \\
\iota & \eta & \epsilon & \delta \\
\kappa & \iota & \theta & \zeta
\end{pmatrix}$$
Problem 3

In the three dimensional Euclidean space, fixate a Cartesian coordinate system, and consider the following.

Let $S$ be a spherical surface of a radius $a \ (> 0)$ centered at the origin. Let $q \in \mathbb{R}^3$ be a position vector of any point $Q$ on $S$, and $n \in \mathbb{R}^3$ be the outward unit normal vector to $S$ at $Q$. $\int_S dS$ represents the surface integral over $S$.

Answer the following questions.

(1) Calculate the following surface integral

$$ I = \int_S \frac{q}{|q|^3} \cdot n \, dS. $$

(2) Let $p \in \mathbb{R}^3$ be a position vector of point $P$ inside $S$ ($|p| < a$). Calculate the following surface integral

$$ \int_S \frac{q - p}{|q - p|^3} \cdot n \, dS, $$

and show that it is the same as $I$ of question (1) above. You may use the Gauss divergence theorem.

(3) Let $B$ be a solid sphere of a radius $b$ centered at the origin. Here, $a > b > 0$. For any point $Q$ on $S$, a vector $F \in \mathbb{R}^3$ is defined as

$$ F = \int_B \frac{q - r}{|q - r|^3} \, dx \, dy \, dz. $$

Here, $r = (x, y, z)^\top \in \mathbb{R}^3$ is a position vector of any point in $B$ ($|r| \leq b$), and $\int_B dx \, dy \, dz$ denotes the volume integral over $B$. $^\top$ denotes the transposition. Calculate the following surface integral

$$ \int_S F \cdot n \, dS. $$

You may use the Gauss divergence theorem.

(4) At any point $Q$ on $S$, the vector $F$ of question (3) above is parallel to $q$. Explain the reason, and calculate $F$. 
Problem 4

Assume that a continuously differentiable real-valued function $f(t)$ on $\mathbb{R}$ satisfies the following equations:

$$
\int_{-\infty}^{\infty} |f(t)|^2 dt = 1, \quad \int_{-\infty}^{\infty} t|f(t)|^2 dt = 0, \quad \lim_{t \to \pm \infty} t|f(t)|^2 = 0.
$$

Also, assume that the derivative $f'(t)$ of $f(t)$ satisfies $\int_{-\infty}^{\infty} |f'(t)|^2 dt < \infty$.

Let $F(\omega)$ be the Fourier transform of $f(t)$ as follows:

$$
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt,
$$

where $i$ denotes the imaginary unit. Answer the following questions. You may use Parseval’s equation

$$
\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega,
$$

where $g(t)$ is a real-valued function on $\mathbb{R}$ that satisfies $\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$ and $G(\omega)$ is the Fourier transform of $g(t)$.

(1) Prove $\int_{-\infty}^{\infty} \omega |F(\omega)|^2 d\omega = 0$, assuming $\int_{-\infty}^{\infty} |\omega||F(\omega)|^2 d\omega < \infty$.

(2) Let $g(t)$ and $h(t)$ be real-valued functions on $\mathbb{R}$, and assume that $\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$ and $\int_{-\infty}^{\infty} |h(t)|^2 dt < \infty$. Let a real number $S$ be $S = \int_{-\infty}^{\infty} g(t)h(t)dt$. Using the fact that $\int_{-\infty}^{\infty} |g(t) + \lambda Sh(t)|^2 dt$ is non-negative for any real number $\lambda$, show the inequality

$$
\int_{-\infty}^{\infty} |g(t)|^2 dt \int_{-\infty}^{\infty} |h(t)|^2 dt \geq S^2.
$$

(3) Prove $\Delta_t \Delta_\omega \geq \frac{1}{2}$, where the real numbers $\Delta_t$ and $\Delta_\omega$ are defined as follows:

$$
\Delta_t = \left(\int_{-\infty}^{\infty} |tf(t)|^2 dt\right)^{\frac{1}{2}}, \quad \Delta_\omega = \left(\int_{-\infty}^{\infty} |\omega F(\omega)|^2 d\omega\right)^{\frac{1}{2}}.
$$

(4) Give an example of $f(t)$ that satisfies $\Delta_t \Delta_\omega = \frac{1}{2}$. 
Let $u(t)$ be a function that maps a non-negative real $t$ into $\mathbb{R}^2$. Consider the solution $u(t) = (x(t), y(t))^\top$ of the following differential equation with an initial condition $u(0) = u_0$:

\[
\begin{align*}
\frac{dx}{dt} &= -xF\left(\sqrt{x^2 + y^2}\right) - y, \\
\frac{dy}{dt} &= -yF\left(\sqrt{x^2 + y^2}\right) + x,
\end{align*}
\] (*)

where the initial value $u_0$ is any element of $\mathbb{R}^2$, $F$ is the function defined by $F(s) = (s-1)(s-2)$, and $\top$ denotes the transposition. Answer the following questions.

1. Changing the variables to polar coordinates, which are defined by $(x, y)^\top = (r \cos \theta, r \sin \theta)^\top$, derive the differential equations for the non-negative real-valued function $r(t)$ and the real-valued function $\theta(t)$.

2. Find the solutions of the differential equations derived in question (1) above, with the initial condition $r(0) = r_0$, $\theta(0) = \theta_0$.

3. When a solution $u(t)$ of the differential equation (*) is a constant function, its constant value $u(t) \in \mathbb{R}^2$ is called an equilibrium. When a solution $u(t)$ is a non-constant and periodic function of $t$, the set $\{u(t) \mid t \geq 0\} \subset \mathbb{R}^2$ is called a periodic orbit.

   Find all the equilibriums and periodic orbits. Express each periodic orbit in the form $\{(x, y)^\top \in \mathbb{R}^2 \mid G(x, y) = 0\}$, where $G(x, y)$ is a polynomial function.

4. Determine a polynomial function $H(x, y)$ such that $H(x(t), y(t))$ is a monotonically non-increasing function of $t$ for any initial value $u_0 \in \mathbb{R}^2$. The function $H(x, y)$ must not be constant.
Problem 6

Let \{X_j; j = 1, 2, 3, \cdots \} be a sequence of mutually independent and identically distributed random variables of non-negative integer. For a random variable \(X\) having a non-negative integer \(k\) with probability \(Pr(X = k)\), the probability generating function \(G_X(z)\) is defined as the expectation of \(z^X\)

\[
G_X(z) = \sum_{k=0}^{\infty} Pr(X = k)z^k \quad \text{for} -1 \leq z \leq 1.
\]

Since the probability generating function \(G_{X_j}(z)\) is independent of \(j\), it is hereafter referred to as \(G(z)\).

Answer the following questions.

(1) Let a random variable \(Y\) have a Poisson distribution \(Po(\gamma)\) defined by

\[
Pr(Y = k) = \frac{\gamma^k e^{-\gamma}}{k!},
\]

where \(\gamma > 0\), and \(k\) is a non-negative integer. Calculate the probability generating function \(G_Y(z)\) for \(Y\).

(2) Represent \(G_{S_n}(z)\) using \(G(z)\), where \(S_n = X_1 + X_2 + \cdots + X_n\) and \(n\) is a non-negative integer. Note that \(S_0 = 0\).

(3) Let \(S_N = X_1 + X_2 + \cdots + X_N\), where \(N\), a random variable of non-negative integer, is independent of any \(X_j\). Show that \(G_{S_N}(z) = G_N(G(z))\).

(4) Let \(N\), a random variable independent of any \(X_j\), have a Poisson distribution \(Po(\lambda)\), where \(\lambda > 0\). Derive the distribution of \(X_j\), which makes \(S_N\) have a Poisson distribution. Assume that \(X_j\) is not constant.
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