2007 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer three problems out of the six problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take the answer sheets and the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

Answer the following questions regarding

$$f(x, y, u, v) = \exp\left(-\frac{(x-u)^2 + (y-v)^2}{2}\right),$$

which is the function on $\mathbb{R}^4$. Here, $\exp p$ denotes e^p for $p \in \mathbb{R}$.

(1) Regard $f(x, y, u, v)$ as the function of u, v. Assume that $|u| \ll 1$ and $|v| \ll 1$. Then, obtain the Taylor series expansion of $f(x, y, u, v)$ for u, v at $(u, v) = (0, 0)$ up to the first order terms. Hereafter, regard the answer to question (1) as the function of $(x, y, u, v) \in \mathbb{R}^4$ and denote it by $g(x, y, u, v)$.

(2) Let $(u, v) = (a \cos\theta, a \sin\theta)$, where $0 < a \ll 1$, $0 \le \theta < 2\pi$. Then, obtain

$$h(x, y) = \int_0^{2\pi} g(x, y, a\cos\theta, a\sin\theta) \cos\theta \; d\theta.$$

(3) Find (x, y) that satisfies $x^2 + y^2 + 2x + 2y = 1$ and maximizes $h(x, y)$.

Problem 2

Assume that A is a real non-singular square matrix. For any A that satisfies each of the following conditions (1)–(6), answer whether A^{-1} satisfies the same condition. Prove it when it is true. Otherwise provide a counterexample with a proof that it does not satisfy the condition. $A^\top$ represents the transpose of A. The (i, j) entry of A is represented as $a_{i,j}$.

(1) Normal matrix, that is, $AA^\top = A^\top A$.

(2) Lower triangular matrix, that is, $a_{i,j} = 0$ for $i < j$.

(3) Unit antidiagonal matrix, that is, $a_{i,n-i+1} = 1$ and other entries are 0, where n is the order of A (i.e., A is an $n \times n$ matrix). An example for $n = 4$ is the following.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(4) Positive matrix, that is, $a_{i,j} > 0$ for all i and j.

(5) Tridiagonal matrix, that is, $a_{i,j} = 0$ for $|i - j| > 1$. An example is the following.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & a_{3,2} & a_{3,3} \end{pmatrix}$$

(6) Persymmetric matrix, that is, $a_{i,j} = a_{n-j+1,n-i+1}$ for all i and j, where n is the order of A. It is symmetric about the antidiagonal (i.e., the diagonal from the top right corner to the bottom left corner). An example for $n = 4$ is the following.

$$A = \begin{pmatrix} \zeta & \delta & \beta & \alpha \\ \theta & \epsilon & \gamma & \beta \\ \iota & \eta & \epsilon & \delta \\ \kappa & \iota & \theta & \zeta \end{pmatrix}$$

Problem 3

In the three dimensional Euclidean space, fixate a Cartesian coordinate system, and consider the following.

Let S be a spherical surface of a radius a (> 0) centered at the origin. Let $\boldsymbol{q} \in \mathbb{R}^3$ be a position vector of any point Q on S, and $\boldsymbol{n} \in \mathbb{R}^3$ be the outward unit normal vector to S at Q. $\int_S dS$ represents the surface integral over S.

Answer the following questions.

(1) Calculate the following surface integral

$$I = \int_S \frac{\boldsymbol{q}}{|\boldsymbol{q}|^3} \cdot \boldsymbol{n} dS.$$

(2) Let $\boldsymbol{p} \in \mathbb{R}^3$ be a position vector of point P inside S $(|\boldsymbol{p}| < a)$. Calculate the following surface integral

$$\int_S \frac{\boldsymbol{q} - \boldsymbol{p}}{|\boldsymbol{q} - \boldsymbol{p}|^3} \cdot \boldsymbol{n} dS,$$

and show that it is the same as I of question (1) above. You may use the Gauss divergence theorem.

(3) Let B be a solid sphere of a radius b centered at the origin. Here, $a > b > 0$. For any point Q on S, a vector $\boldsymbol{F} \in \mathbb{R}^3$ is defined as

$$\boldsymbol{F} = \int_B \frac{\boldsymbol{q} - \boldsymbol{r}}{|\boldsymbol{q} - \boldsymbol{r}|^3} dxdydz.$$

Here, $\boldsymbol{r} = (x, y, z)^\top \in \mathbb{R}^3$ is a position vector of any point in B $(|\boldsymbol{r}| \leq b)$, and $\int_B dxdydz$ denotes the volume integral over B. $\top$ denotes the transposition. Calculate the following surface integral

$$\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS.$$

You may use the Gauss divergence theorem.

(4) At any point Q on S, the vector $\boldsymbol{F}$ of question (3) above is parallel to $\boldsymbol{q}$. Explain the reason, and calculate $\boldsymbol{F}$.

Problem 4

Assume that a continuously differentiable real-valued function $f(t)$ on $\mathbb{R}$ satisfies the following equations:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = 1, \qquad \int_{-\infty}^{\infty} t|f(t)|^2 dt = 0, \qquad \lim_{t \to \pm\infty} t|f(t)|^2 = 0.$$

Also, assume that the derivative $f'(t)$ of $f(t)$ satisfies $\displaystyle\int_{-\infty}^{\infty} |f'(t)|^2 dt < \infty$. Let $F(\omega)$ be the Fourier transform of $f(t)$ as follows:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt,$$

where i denotes the imaginary unit. Answer the following questions. You may use Parseval's equation

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega,$$

where $g(t)$ is a real-valued function on $\mathbb{R}$ that satisfies $\displaystyle\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$ and $G(\omega)$ is the Fourier transform of $g(t)$.

(1) Prove $\displaystyle\int_{-\infty}^{\infty} \omega |F(\omega)|^2 d\omega = 0$, assuming $\displaystyle\int_{-\infty}^{\infty} |\omega| |F(\omega)|^2 d\omega < \infty$.

(2) Let $g(t)$ and $h(t)$ be real-valued functions on $\mathbb{R}$, and assume that $\displaystyle\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$ and $\displaystyle\int_{-\infty}^{\infty} |h(t)|^2 dt < \infty$. Let a real number S be $S = \displaystyle\int_{-\infty}^{\infty} g(t)h(t) dt$. Using the fact that $\displaystyle\int_{-\infty}^{\infty} |g(t) + \lambda S h(t)|^2 dt$ is non-negative for any real number λ, show the inequality

$$\int_{-\infty}^{\infty} |g(t)|^2 dt \int_{-\infty}^{\infty} |h(t)|^2 dt \geq S^2.$$

(3) Prove $\Delta_t \Delta_\omega \geq \dfrac{1}{2}$, where the real numbers Δ_t and Δ_ω are defined as follows:

$$\Delta_t = \left(\int_{-\infty}^{\infty} |t f(t)|^2 dt \right)^{\frac{1}{2}}, \qquad \Delta_\omega = \left(\int_{-\infty}^{\infty} |\omega F(\omega)|^2 d\omega \right)^{\frac{1}{2}}.$$

(4) Give an example of $f(t)$ that satisfies $\Delta_t \Delta_\omega = \dfrac{1}{2}$.

Problem 5

Let $\boldsymbol{u}(t)$ be a function that maps a non-negative real t into $\mathbb{R}^2$. Consider the solution $\boldsymbol{u}(t) = (x(t), y(t))^\top$ of the following differential equation with an initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$:

$$\begin{aligned} \frac{dx}{dt} &= -xF\left(\sqrt{x^2+y^2}\right) - y, \\ \frac{dy}{dt} &= -yF\left(\sqrt{x^2+y^2}\right) + x, \end{aligned} \tag{$*$}$$

where the initial value $\boldsymbol{u}_0$ is any element of $\mathbb{R}^2$, F is the function defined by $F(s) = (s-1)(s-2)$, and $\top$ denotes the transposition. Answer the following questions.

(1) Changing the variables to polar coordinates, which are defined by $(x, y)^\top = (r\cos\theta, r\sin\theta)^\top$, derive the differential equations for the non-negative real-valued function $r(t)$ and the real-valued function $\theta(t)$.

(2) Find the solutions of the differential equations derived in question (1) above, with the initial condition $r(0) = r_0$, $\theta(0) = \theta_0$.

(3) When a solution $\boldsymbol{u}(t)$ of the differential equation $(*)$ is a constant function, its constant value $\boldsymbol{u}(t) \in \mathbb{R}^2$ is called an equilibrium. When a solution $\boldsymbol{u}(t)$ is a non-constant and periodic function of t, the set $\{\boldsymbol{u}(t) \mid t \geq 0\} \subset \mathbb{R}^2$ is called a periodic orbit.

Find all the equilibriums and periodic orbits. Express each periodic orbit in the form $\{(x, y)^\top \in \mathbb{R}^2 \mid G(x, y) = 0\}$, where $G(x, y)$ is a polynomial function.

(4) Determine a polynomial function $H(x, y)$ such that $H(x(t), y(t))$ is a monotonically non-increasing function of t for any initial value $\boldsymbol{u}_0 \in \mathbb{R}^2$. The function $H(x, y)$ must not be constant.

Problem 6

Let $\{X_j; j = 1, 2, 3, \cdots\}$ be a sequence of mutually independent and identically distributed random variables of non-negative integer. For a random variable X having a non-negative integer k with probability $Pr(X = k)$, the probability generating function $G_X(z)$ is defined as the expectation of z^X

$$G_X(z) = \sum_{k=0}^{\infty} Pr(X = k) z^k \qquad \text{for } -1 \leq z \leq 1.$$

Since the probability generating function $G_{X_j}(z)$ is independent of j, it is hereafter referred to as $G(z)$.

Answer the following questions.

(1) Let a random variable Y have a Poisson distribution $Po(\gamma)$ defined by

$$Pr(Y = k) = \frac{\gamma^k e^{-\gamma}}{k!},$$

where $\gamma > 0$, and k is a non-negative integer. Calculate the probability generating function $G_Y(z)$ for Y.

(2) Represent $G_{S_n}(z)$ using $G(z)$, where $S_n = X_1 + X_2 + \cdots + X_n$ and n is a non-negative integer. Note that $S_0 = 0$.

(3) Let $S_N = X_1 + X_2 + \cdots + X_N$, where N, a random variable of non-negative integer, is independent of any X_j. Show that $G_{S_N}(z) = G_N(G(z))$.

(4) Let N, a random variable independent of any X_j, have a Poisson distribution $Po(\lambda)$, where $\lambda > 0$. Derive the distribution of X_j, which makes S_N have a Poisson distribution. Assume that X_j is not constant.

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