2006 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer three problems out of the six problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols unrelated to the answer on it.
8. Do not take the answer sheets and the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

Consider a set X whose size is finite and a map $f : X \to X$. For each non-negative integer $i = 0, 1, \ldots$, let f^i be the map obtained from f by composing it i times; that is, the map defined as follows.

$$\begin{array}{rcl} f^0(x) & = & x \\ f^i(x) & = & f(f^{i-1}(x)) \quad (i \geq 1) \end{array}$$

Given a set $Y \subseteq X$ and a map $g : X \to X$, the set Y is said to be a *fixed point* of g when Y satisfies

$$g(Y) = Y.$$

Here, we define $g(Y) = \{g(y) \mid y \in Y\}$.

Answer the following questions.

(1) For each non-negative integer $i = 0, 1, \ldots$, define the set

$$A_i \quad = \quad f^i(X).$$

Prove that there exists k such that $A_k = A_{k+1}$.

(2) Let k^* be one of the k's that satisfy the condition in the question (1). Prove that A_{k^*} contains all fixed points of f.

(3) For an element x of X, define the set

$$B(x) = \{f^i(x) \mid i \geq k^*\}.$$

Prove that $B(x)$ is a fixed point of f.

(4) Prove $A_{k^*} = \bigcup_{x \in X} B(x)$.

Problem 2

Let $a_1(t)$ and $a_2(t)$ be continuously differentiable functions that map real numbers to real numbers. Let $\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix}$ be denoted by $\boldsymbol{a}(t)$. Consider the following initial-value problem on a differential equation in $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = -\boldsymbol{a}(t)\boldsymbol{a}(t)^\top \boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{2.1}$$

Here, $\boldsymbol{a}(t)^\top$ stands for the transpose of the vector $\boldsymbol{a}(t)$. Answer the following questions.

(1) Suppose that $a_1(t) = 1$ and $a_2(t) = 0$ for any real number t. Prove that the solution $\boldsymbol{x}(t)$ of the initial-value problem (2.1) does not converge to the zero vector in the limit $t \to \infty$.

(2) Suppose that we choose $p_1(t)$ and $p_2(t)$ to be some continuously differentiable functions that map real numbers to real numbers, satisfying $p_1(t)^2 + p_2(t)^2 = 1$. Using the variable transformation

$$\boldsymbol{y}(t) = P(t)\boldsymbol{x}(t)$$

based on the matrix

$$P(t) = \begin{bmatrix} p_1(t) & p_2(t) \\ -p_2(t) & p_1(t) \end{bmatrix},$$

rewrite the initial-value problem (2.1) into an initial-value problem on a differential equation in $\boldsymbol{y}(t)$.

(3) Suppose that $a_1(t) = \cos t$ and $a_2(t) = \sin t$ for any real number t. Prove that the solution $\boldsymbol{x}(t)$ of the initial-value problem (2.1) converges to the zero vector in the limit $t \to \infty$.

(4) Considering the results of (1) and (3), conjecture what property on $\boldsymbol{a}(t)$ guarantees that the solution $\boldsymbol{x}(t)$ of the initial-value problem (2.1) converges to the zero vector in the limit $t \to \infty$. Moreover, explain why you conjecture in that way. There is no need to prove the conjecture.

Problem 3

Consider the contour integral of the following complex function $f(z)$ along the path $C(= C_1 + C_2 + C_3 + C_4 + C_5 + C_6)$ as shown in Figure 3.1. Answer the following questions.

$$f(z) = \frac{e^{iz}}{z - \pi}$$

(1) Determine all the poles of $f(z)$, and the residues at the poles.

(2) Calculate the following contour integral.

$$\int_C f(z)dz$$

(3) Calculate the following integral of $f(z)$ along the path $C_4 + C_5 + C_6$ in the limit $X_1, Y, X_2 \to +\infty$.

$$\lim_{X_1, Y, X_2 \to +\infty} \int_{C_4 + C_5 + C_6} f(z)dz$$

(4) Calculate the following integral of $f(z)$ along the path C_2 in the limit $\rho \to 0$.

$$\lim_{\rho \to 0} \int_{C_2} f(z)dz$$

(5) Calculate the following definite integral.

$$\lim_{X_1, X_2 \to +\infty,\ \rho \to 0} \left(\int_{-X_2}^{\pi - \rho} \frac{\sin x}{x - \pi} dx + \int_{\pi + \rho}^{X_1} \frac{\sin x}{x - \pi} dx \right)$$

Here, the above integral is called the principal value of the following integral.

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x - \pi} dx$$

(6) Calculate the following definite integral when a is a real number and $a \neq 0$.

$$\lim_{X_1, X_2 \to +\infty,\ \rho \to 0} \left(\int_{-X_2}^{\pi - \rho} \frac{\sin ax}{x - \pi} dx + \int_{\pi + \rho}^{X_1} \frac{\sin ax}{x - \pi} dx \right)$$

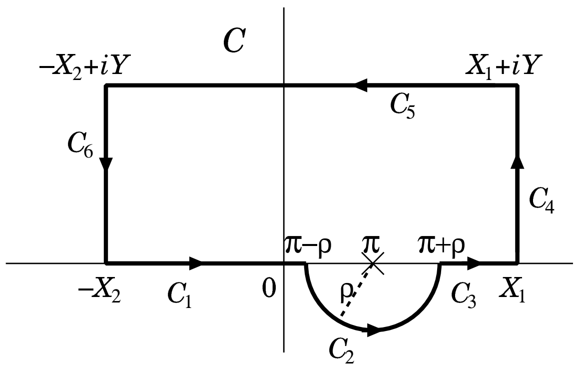

Figure 3.1

Problem 4

Consider a point P, starting from the origin O at time $t = 0$ and moving on a number line. The point P moves by $+1$ with probability $1/2$, and moves by -1 with probability $1/2$, at every unit time. Answer the following questions.

(1) Calculate all the possible positions of the point P at time $t = 2$ and their corresponding probabilities.

(2) Calculate the probability $p(x, t)$ with which the position of the point P is x at time t.

(3) Calculate the expectation and the variance of the position x of the point P at time t.

(4) Show that $p(x, t)$ can be approximated by

$$\frac{2}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}$$

when t is sufficiently large, t and x are both odd numbers or both even numbers, and $|x| \leq \sqrt{t}$. Explain the meaning of this approximation. You may use $n! \simeq \sqrt{2\pi n} \cdot n^n \cdot e^{-n}$ (Stirling's formula) when n is sufficiently large, and $\log_e(1 + a) \simeq a - \frac{1}{2}a^2$ when a is sufficiently small.

Problem 5

Let X be an $n \times n$ real symmetric matrix. In particular, X is said to be positive semidefinite if $\boldsymbol{v}^\top X \boldsymbol{v} \geq 0$ holds for any n-dimensional real vector $\boldsymbol{v}$. Here, the symbol $^\top$ stands for the transpose of a vector or a matrix. Moreover, for an $n \times n$ real symmetric matrix Y, let $\operatorname{tr} XY$ mean the sum of all the diagonal elements of the matrix XY. Answer the following questions.

(1) For an $n \times n$ real symmetric matrix X, it is known that one can make the matrix $U^\top XU$ have the form of a diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

by choosing an $n \times n$ orthogonal matrix U appropriately. Let the vector $\boldsymbol{u}_i$ be the ith column of such U. Prove that $\boldsymbol{u}_i$ is an eigenvector of X and λ_i is the corresponding eigenvalue for each $i = 1, 2, \ldots, n$.

(2) Prove that each eigenvalue of X is a nonnegative real number if and only if X is positive semidefinite.

(3) Prove that there exists a matrix A satisfying $X = A^\top A$ if and only if X is positive semidefinite.

(4) Prove that $\operatorname{tr} XY \geq 0$ holds if both X and Y are positive semidefinite.

(5) Prove that $\operatorname{tr} XY \geq 0$ holds for any $n \times n$ positive semidefinite matrix Y if and only if X is positive semidefinite.

Problem 6

Answer the following questions.

(1) Consider an infinite collection of parallel lines forming a grating with a uniform line separation distance of 2 units in a plane. When a square with each side of length $\sqrt{2}$ units is placed at random, in terms of the position and the direction, as shown in Figure 6.1, calculate the probability for the square to intersect with at least one of the lines.

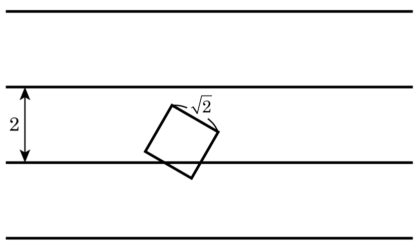

Figure 6.1

(2) Consider an infinite collection of parallel lines forming a grating with a uniform line separation distance of 2 units in a plane in the same way as in (1). When a rectangle with each longer side of length $\sqrt{3}$ units and each shorter side of length 1 unit is placed at random, in terms of the position and the direction, as shown in Figure 6.2, calculate the probability for the rectangle to intersect with at least one of the lines.

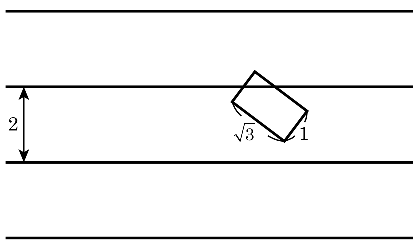

Figure 6.2

(3) Consider an infinite collection of orthogonal grid lines with a uniform line separation distance of 2 units in a plane. When a rectangle with each longer side of length $\sqrt{3}$ units and each shorter side of length 1 unit is placed at random, in terms of the position and the direction, as shown in Figure 6.3, calculate the probability for the rectangle to intersect with at least one of the lines.

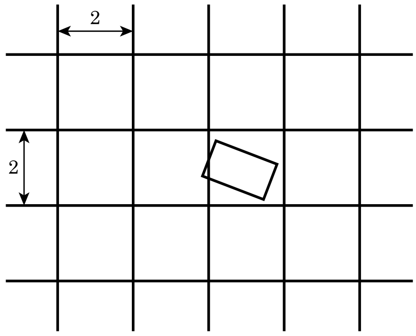

Figure 6.3

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