# 2005 School Year

# Graduate School
# Entrance Examination Problem Booklet

# Mathematics

10:00 a.m. to 12:30 a.m.

### Instruction

Answers should be written in **Japanese** or **English**.

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
4. Fill the designated blanks at the top of each answer sheet with your examinee's number and the number you are to answer.
5. Answer three problems out of the following six problems.
6. The blank pages are provided for making draft. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols unrelated to the answer on it.
8. Do not take the answer sheets and the problem booklet out of the examination room.

| | |
|---|---|
| Examinee's number | No. |

Fill this box with your examinee's number.

(blank page for draft)

## Problem 1

Consider a cubic die with the numbers from one to six on its faces. The die is rotated 90 degrees with one of four edges of the bottom face as an axis while time $t$ advances 1. An example of the rotation operation is shown in Fig. 1. The probabilities of selecting each edge from four edges are equal. On the die, the faces with the number from two to five share edges with the face with the number one, and the face with the number six is the opposite of the face with the number one. Answer the following questions.

(1) Let $p_1(t)$ be the probability that the number one appears on the top face at time $t$, $p_2(t)$ be the probability that either of the numbers from two to five appears on the top face, and $p_3(t)$ be the probability that the number six appears on the top face. Obtain the matrix $A$ where

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \\ p_3(t+1) \end{bmatrix} = A \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}.$$

(2) Calculate the eigenvalues and the eigenvectors of the matrix $A$.

(3) Calculate $A^n$ where $n$ is a positive integer.

(4) Calculate $p_1(t)$ under the initial condition that the number one appears on the top face at the time $t = 0$.

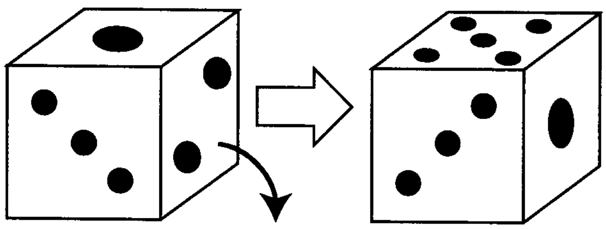

Fig. 1 Rotation operation of a cubic die.

## Problem 2

Let $T$ be a set. Let $A$, $B$, $C$, and $D$ be non-empty, proper subsets of $T$. Let $f(E)$ be the image of a set $E$ by a mapping $f$. Consider the following three conditions.

(i) $B \subseteq D$.

(ii) $C \subseteq A$.

(iii) For any mapping $f$ from $T$ to $T$, if $f(A) \subseteq B$, then $f(C) \subseteq D$.

To prove that the condition "(i) and (ii)" is equivalent to the condition (iii), answer the following questions.

(1) Prove that "(i) and (ii)" implies (iii).

(2) Prove that (iii) implies (i).

(3) Prove that (iii) implies (ii).

## Problem 3

Let $A_1, A_2, A_3, \ldots$ be independent uniform random variables over the set of integers $\{1, 2, \ldots, m\}$ and $J$ be the minimum index satisfying $A_J \neq A_{J+1}$. Let $J_1, J_2, \ldots, J_n$ be independent, identically distributed random variables whose probability density functions are identical with that of $J$. Let $L_n = \min\{J_1, J_2, \ldots, J_n\}$ and $U_n = \max\{J_1, J_2, \ldots, J_n\}$.

Answer the following questions.

(1) Obtain the expected value of $J$.

(2) Obtain the expected value of $L_n$.

(3) Given a positive integer $k$, obtain the probability $\mathrm{Prob}[U_n = k]$.

(4) Find $\lim_{k \to \infty} \mathrm{Prob}[U_{m^k} = k]$.

## Problem 4

Let $n$ be a positive integer. Answer the following questions about the volume of an $n$-dimensional sphere in the $n$-dimensional Euclidean space given by

$$V_n(a) = \int\int\cdots\int_{x_1^2+x_2^2+\cdots+x_n^2 \le a^2} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n,$$

where $a$ is the radius of the sphere.

(1) Show that the volume $V_n(a)$ satisfies a recursion formula

$$V_n(a) = aI_nV_{n-1}(a),$$

where

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n\theta \mathrm{d}\theta.$$

(2) Describe the volume $V_n(a)$ as a recursion formula

$$V_n(a) = c_nV_{n-2}(a),$$

and obtain the coefficient $c_n$.

(3) Obtain $\lim_{n\to\infty} V_n(a)$.

## Problem 5

For an arbitrary complex number $\alpha$ whose absolute value is not equal to 1, let a function of a real variable $t$ be

$$\psi(t, \alpha) = \frac{\mathrm{d}}{\mathrm{d}t} \log(e^{2\pi jt} - \alpha),$$

where $j$ is the imaginary unit. Answer the following questions.

(1) Obtain the actual form of $\psi(t, \alpha)$.

(2) Describe

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\left( \frac{1}{2} - e^{2\pi jt} + e^{4\pi jt} \right)$$

using the function $\psi$.

(3) Introducing $z = e^{2\pi jt}$, calculate the integral

$$\int_0^1 \psi(t, \alpha)\mathrm{d}t.$$

(4) Introducing $z = e^{2\pi jt}$, calculate the integral

$$\int_0^1 \psi(t, \alpha_1)\psi^*(t + t_0, \alpha_2)\mathrm{d}t,$$

where "*" denotes the complex conjugate, $\alpha_1, \alpha_2$ are arbitrary complex numbers whose absolute values are not equal to 1, and $t_0$ is an arbitrary real number.

## Problem 6

Let $P=\{2,3,5,7,\ldots\}$ be the set of all prime numbers. Consider the following system of differential equations for $x$ and $y$ with respect to time $t$ ($\geq 0$):

$$\begin{cases} 2\dot{x} - ay &= \sin(2t), \\ 2ax + \dot{y} &= \cos(2t), \end{cases}$$

where $\dot{x} = \dfrac{\mathrm{d}x}{\mathrm{d}t}$, $\dot{y} = \dfrac{\mathrm{d}y}{\mathrm{d}t}$, and $a \in P$ is a constant. Let $x = x_a(t)$ and $y = y_a(t)$ be the solutions to the system of differential equations with the initial conditions $x(0) = y(0) = 0$.

(1) Obtain $x_a(t)$ and $y_a(t)$ for $a > 2$.

(2) Obtain $x_2(t)$ and $y_2(t)$.

(3) Using the fact that the series

$$\sum_{a \in P} \frac{1}{a} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

diverges, prove that the series

$$\sum_{a \in P} x_a(\pi) = x_2(\pi) + x_3(\pi) + x_5(\pi) + x_7(\pi) + \cdots$$

diverges.

(blank page for draft)

(blank page for draft)