Problem 1

For a real number q such that $0 < q < 1$, let A be the matrix defined by

$$A = \begin{pmatrix} q & 1-q \\ q^2 & 1-q^2 \end{pmatrix}.$$

Answer the following questions.

(1) Calculate the eigenvalues and the eigenvectors of the matrix A.

(2) Calculate $\exp(A)$ defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

(3) Find q which maximizes the determinant of $\exp(A)$ and its maximum value.

Problem 2

Let $f(x)$ be the function defined by

$$f(x) = \frac{\cos(2x) - \cos(x)}{\cos(2x) + 2\cos(x) + 1}$$

for real numbers x such that $\cos(2x) + 2\cos(x) + 1 \neq 0$.

Answer the following questions.

(1) Draw a rough shape of the graph of the function $f(x)$ for $\pi < x < 2\pi$.

(2) By using the above function $f(x)$, we define a sequence $\{a_n\}$ in the following recursive procedure.

1. $a_1 = 4$.
2. When the value of a_n is defined, let $t_n = a_n + 2^{-n}$ and

$$a_{n+1} = \begin{cases} a_n & \text{if } f(t_n) \geq 0, \\ t_n & \text{if } f(t_n) < 0. \end{cases}$$

Let $\{b_n\}$ be the sequence defined by

$$b_n = 2^n(a_{n+1} - a_n).$$

Obtain the first 10 terms $b_1, b_2, \ldots, b_{10}$ of the sequence $\{b_n\}$. You may use $\pi = 3.14159\cdots$.

Problem 3

For a set X, 2^X denotes the set of all subsets of X. Prove that for any set X there exists no mapping $f : X \to 2^X$ satisfying the following condition:

Condition (C) For any element A of the set 2^X, there exists an element a of the set X such that $f(a) = A$.

Problem 4

Let us consider solving the following voting problem:

"The purpose of the voting is to elect one out of the two candidates P and Q. The candidates P and Q got p votes and q votes, respectively, and the candidate P was elected. There was no blank ballot or no invalid ballot. The total number of votes was $p+q$. Obtain the probability that the candidate P gets more votes than the candidate Q at any time in the process of counting the votes one by one in a random order."

Let $\mathbf{Z}$ be the set of all integers, and $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$ be the set of all points with integer coordinates in the (x, y) plane. Let a and b be integers such that $a < b$. An integer-valued function s defined on the subset $\{a, a+1, \ldots, b-1, b\}$ of the set $\mathbf{Z}$ is called a path from point $(a, s(a))$ to point $(b, s(b))$ if

$$|s(k) - s(k-1)| = 1$$

is satisfied for any $k = a+1, a+2, \ldots, b-1, b$. For any two points $\mathrm{A} = (a, \alpha)$, $\mathrm{B} = (b, \beta)$ $(a < b)$ in the set $\mathbf{Z}^2$, we denote by $\Omega(\mathrm{A}, \mathrm{B})$ the set of all paths from A to B.

For $k = 0, 1, \ldots, p+q$, let $s(k)$ be the number obtained by subtracting the number of the votes Q gets from the number of the votes P gets when the first k votes are counted. Then, the function s can be considered a path from the point $\mathrm{O} = (0, 0)$ to the point $\mathrm{D} = (p+q, p-q)$. Hence, the set of all possible paths corresponding to the transition process of the difference of the votes for P and Q is represented by $\Omega(\mathrm{O}, \mathrm{D})$. Let V be the set of all possible paths corresponding to the transition process of the difference of the votes for P and Q such that P always gets more votes than Q. Then, V can be represented by

$$V = \{s \in \Omega(\mathrm{O}, \mathrm{D}) \mid s(k) > 0, k = 1, 2, \ldots, p+q\}.$$

Answer the following questions.

(1) Prove that for any point $\mathrm{B} = (b, \beta)$ $(b > 0)$ in the set $\mathbf{Z}^2$, the set $\Omega(\mathrm{O}, \mathrm{B})$ is nonempty if and only if $b + \beta \geq 0$, $b - \beta \geq 0$ and $b + \beta$ is even.

(2) For any finite set S, let $n(S)$ denote the number of elements of S. Prove that if the set $\Omega(\mathrm{O}, \mathrm{B})$ is nonempty for a point $\mathrm{B} = (b, \beta)$ $(b > 0)$ in the set $\mathbf{Z}^2$, then

$$n(\Omega(\mathrm{O}, \mathrm{B})) = \binom{b}{\frac{b+\beta}{2}}$$

holds, where $\binom{i}{j}$ represents the number of combinations of choosing j objects out of i distinguishable objects.

(3) Let A $= (a, \alpha)$, B $= (b, \beta)$ in the set $\mathbf{Z}^2$ satisfy $0 \leq a < b, \alpha > 0$ and $\beta > 0$. We call the point A$' = (a, -\alpha)$ in the set $\mathbf{Z}^2$ the reflection point of the point A with respect to the x axis. Define a subset W of the set $\Omega(\mathrm{A}, \mathrm{B})$ by

$$W = \{s \in \Omega(\mathrm{A}, \mathrm{B}) \mid \text{The path } s \text{ has at least one common point with the } x \text{ axis}\}.$$

Prove that the following holds:

$$n(W) = n(\Omega(\mathrm{A}', \mathrm{B})).$$

(4) Let C $= (1, 1)$, and let C$' = (1, -1)$ be the reflection point of the point C with respect to the x axis. Prove that the following holds:

$$n(V) = n(\Omega(\mathrm{C}, \mathrm{D})) - n(\Omega(\mathrm{C}', \mathrm{D})).$$

(5) Obtain the probability requested in the voting problem stated at the beginning.

Problem 5

Suppose that a complex function $f(z) = u(z) + \mathrm{i}v(z)$ of a complex variable z is holomorphic and is not constant in the open disk $D = \{z \mid |z - z_0| < r_0\}$ centered at the point z_0 in the complex plane with the radius r_0, where $u(z)$ and $v(z)$ are the real and imaginary parts, respectively, of $f(z)$.

Answer the following questions.

(1) Prove from Cauchy's integral formula, that the equations

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta, \quad v(z) = \frac{1}{2\pi} \int_0^{2\pi} v(z + r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta$$

hold for any $z \in D$ and real number r such that $0 < r < r_0 - |z - z_0|$.

(2) Prove that neither the function u nor v takes the maximum value in the open disk D. Prove also that neither u nor v takes the minimum value in the open disk D.

(3) Let us consider a point z in the open disk D such that the derivative $f'(z)$ of f at z is not equal to 0. Prove that the curve passing through z along which u is constant and the curve passing through z along which v is constant cross at z orthogonally to each other.

Problem 6

Let A and B be distinct real constants. Suppose that $u(t,x)$ is a sufficiently smooth function defined for $t \geq 0, 0 \leq x \leq 1$, and is specified by the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (t > 0,\ 0 < x < 1) \tag{6.1}$$

together with the boundary conditions

$$u(t,0) = A, \quad u(t,1) = B \qquad (t \geq 0) \tag{6.2}$$

and the initial condition

$$u(0,x) = \sin(3\pi x) + (B - A)x + A \qquad (0 \leq x \leq 1). \tag{6.3}$$

Answer the following questions.

(1) Suppose that $u(t,x)$ represents the temperature at time t and point x of a sufficiently thin rod placed along the x axis. Give a physical interpretation of the equation (6.1) and the boundary conditions (6.2).

(2) According to the interpretation given in (1), guess the limit $f(x)$ of the function $u(t,x)$ as $t \to \infty$.

(3) Suppose that $u(t,x)$ is represented as

$$u(t,x) = w(t,x) + f(x) \tag{6.4}$$

with the function $f(x)$ guessed in (2). Give the equation, the boundary conditions, and the initial condition satisfied by $w(t,x)$.

(4) Assuming that $w(t,x)$ is represented as the product $P(t)Q(x)$ of a function $P(t)$ of t and a function $Q(x)$ of x, find the expression of $w(t,x)$.

(5) Find the expression of $u(t,x)$ using the result obtained in (4).