Problem 1

For a real number $q$ such that $0 < q < 1$, let $A$ be the matrix defined by

$$A = \begin{pmatrix} q & 1 - q \\ q^2 & 1 - q^2 \end{pmatrix}.$$ 

Answer the following questions.

(1) Calculate the eigenvalues and the eigenvectors of the matrix $A$.

(2) Calculate $\exp(A)$ defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$ 

(3) Find $q$ which maximizes the determinant of $\exp(A)$ and its maximum value.
Problem 2

Let \( f(x) \) be the function defined by
\[
f(x) = \frac{\cos(2x) - \cos(x)}{\cos(2x) + 2 \cos(x) + 1}
\]
for real numbers \( x \) such that \( \cos(2x) + 2 \cos(x) + 1 \neq 0 \).

Answer the following questions.

1. Draw a rough shape of the graph of the function \( f(x) \) for \( \pi < x < 2\pi \).
2. By using the above function \( f(x) \), we define a sequence \( \{a_n\} \) in the following recursive procedure.
   1. \( a_1 = 4 \).
   2. When the value of \( a_n \) is defined, let \( t_n = a_n + 2^{-n} \) and
      \[
a_{n+1} = \begin{cases} 
a_n & \text{if } f(t_n) \geq 0, \\
t_n & \text{if } f(t_n) < 0.
\end{cases}
\]

Let \( \{b_n\} \) be the sequence defined by
\[
b_n = 2^n(a_{n+1} - a_n).
\]

Obtain the first 10 terms \( b_1, b_2, \ldots, b_{10} \) of the sequence \( \{b_n\} \). You may use \( \pi = 3.14159 \cdots \).
Problem 3

For a set $X$, $2^X$ denotes the set of all subsets of $X$. Prove that for any set $X$ there exists no mapping $f : X \rightarrow 2^X$ satisfying the following condition:

Condition (C) For any element $A$ of the set $2^X$, there exists an element $a$ of the set $X$ such that $f(a) = A$. 
Problem 4

Let us consider solving the following voting problem:

"The purpose of the voting is to elect one out of the two candidates P and Q. The candidates P and Q got p votes and q votes, respectively, and the candidate P was elected. There was no blank ballot or no invalid ballot. The total number of votes was p + q. Obtain the probability that the candidate P gets more votes than the candidate Q at any time in the process of counting the votes one by one in a random order."

Let $\mathbb{Z}$ be the set of all integers, and $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ be the set of all points with integer coordinates in the $(x,y)$ plane. Let $a$ and $b$ be integers such that $a < b$. An integer-valued function $s$ defined on the subset $\{a, a+1, \ldots, b-1, b\}$ of the set $\mathbb{Z}$ is called a path from point $(a, s(a))$ to point $(b, s(b))$ if

$$|s(k) - s(k-1)| = 1$$

is satisfied for any $k = a+1, a+2, \ldots, b-1, b$. For any two points $A = (a, \alpha)$, $B = (b, \beta)$ $(a < b)$ in the set $\mathbb{Z}^2$, we denote by $\Omega(A, B)$ the set of all paths from $A$ to $B$.

For $k = 0, 1, \ldots, p + q$, let $s(k)$ be the number obtained by subtracting the number of the votes Q gets from the number of the votes P gets when the first $k$ votes are counted. Then, the function $s$ can be considered a path from the point $O = (0, 0)$ to the point $D = (p + q, p - q)$. Hence, the set of all possible paths corresponding to the transition process of the difference of the votes for P and Q is represented by $\Omega(O, D)$. Let $V$ be the set of all possible paths corresponding to the transition process of the difference of the votes for P and Q such that P always gets more votes than Q. Then, $V$ can be represented by

$$V = \{s \in \Omega(O, D) \mid s(k) > 0, k = 1, 2, \ldots, p + q\}.$$

Answer the following questions.

(1) Prove that for any point $B = (b, \beta)$ $(b > 0)$ in the set $\mathbb{Z}^2$, the set $\Omega(O, B)$ is nonempty if and only if $b + \beta \geq 0$, $b - \beta \geq 0$ and $b - \beta$ is even.

(2) For any finite set $S$, let $n(S)$ denote the number of elements of $S$. Prove that if the set $\Omega(O, B)$ is nonempty for a point $B = (b, \beta)$ $(b > 0)$ in the set $\mathbb{Z}^2$, then

$$n(\Omega(O, B)) = \binom{b}{\frac{b + \beta}{2}}$$

holds, where $\binom{j}{i}$ represents the number of combinations of choosing $j$ objects out of $i$ distinguishable objects.
(3) Let $A = (a, \alpha), B = (b, \beta)$ in the set $\mathbb{Z}^2$ satisfy $0 \leq a < b, \alpha > 0$ and
$\beta > 0$. We call the point $A' = (a, -\alpha)$ in the set $\mathbb{Z}^2$ the reflection point
of the point $A$ with respect to the $x$ axis. Define a subset $W$ of the set
$\Omega(A, B)$ by

$$W = \{ s \in \Omega(A, B) \mid \text{The path } s \text{ has at least}
\quad \text{one common point with the } x \text{ axis}\}. $$

Prove that the following holds:

$$n(W) = n(\Omega(A', B)).$$

(4) Let $C = (1, 1)$, and let $C' = (1, -1)$ be the reflection point of the point
$C$ with respect to the $x$ axis. Prove that the following holds:

$$n(V) = n(\Omega(C, D)) - n(\Omega(C', D)).$$

(5) Obtain the probability requested in the voting problem stated at the
beginning.
Problem 5

Suppose that a complex function \( f(z) = u(z) + iv(z) \) of a complex variable \( z \) is holomorphic and is not constant in the open disk \( D = \{ z \mid |z - z_0| < r_0 \} \) centered at the point \( z_0 \) in the complex plane with the radius \( r_0 \), where \( u(z) \) and \( v(z) \) are the real and imaginary parts, respectively, of \( f(z) \).

Answer the following questions.

(1) Prove from Cauchy's integral formula, that the equations

\[
    u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \, d\theta, \quad v(z) = \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta}) \, d\theta
\]

hold for any \( z \in D \) and real number \( r \) such that \( 0 < r < r_0 - |z - z_0| \).

(2) Prove that neither the function \( u \) nor \( v \) takes the maximum value in the open disk \( D \). Prove also that neither \( u \) nor \( v \) takes the minimum value in the open disk \( D \).

(3) Let us consider a point \( z \) in the open disk \( D \) such that the derivative \( f'(z) \) of \( f \) at \( z \) is not equal to 0. Prove that the curve passing through \( z \) along which \( u \) is constant and the curve passing through \( z \) along which \( v \) is constant cross at \( z \) orthogonally to each other.
Problem 6

Let $A$ and $B$ be distinct real constants. Suppose that $u(t, x)$ is a sufficiently smooth function defined for $t \geq 0, 0 \leq x \leq 1$, and is specified by the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (t > 0, 0 < x < 1) \quad (6.1)$$

together with the boundary conditions

$$u(t, 0) = A, \quad u(t, 1) = B \quad (t \geq 0) \quad (6.2)$$

and the initial condition

$$u(0, x) = \sin(3\pi x) + (B - A)x + A \quad (0 \leq x \leq 1). \quad (6.3)$$

Answer the following questions.

(1) Suppose that $u(t, x)$ represents the temperature at time $t$ and point $x$ of a sufficiently thin rod placed along the $x$ axis. Give a physical interpretation of the equation (6.1) and the boundary conditions (6.2).

(2) According to the interpretation given in (1), guess the limit $f(x)$ of the function $u(t, x)$ as $t \to \infty$.

(3) Suppose that $u(t, x)$ is represented as

$$u(t, x) = w(t, x) + f(x) \quad (6.4)$$

with the function $f(x)$ guessed in (2). Give the equation, the boundary conditions, and the initial condition satisfied by $u(t, x)$.

(4) Assuming that $w(t, x)$ is represented as the product $P(t)Q(x)$ of a function $P(t)$ of $t$ and a function $Q(x)$ of $x$, find the expression of $w(t, x)$.

(5) Find the expression of $u(t, x)$ using the result obtained in (4).