平成14(2002)年度大学院入試英文説明書(数学)

# Instruction (Mathematics)

Answers should be written in **Japanese** or **English**.

(1) Do not open this problem booklet until the start of the examination is announced.

(2) If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.

(3) You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.

(4) Fill the designated blanks at the top of each answer sheet with your examinee's number and the number of the problem you are to answer.

(5) Answer three problems out of the following six problems.

(6) The blank pages are provided for making draft. Do not detach them from this problem booklet.

(7) An answer sheet is regarded as invalid if you write marks and/or symbols unrelated to the answer on it.

(8) Do not take the answer sheets and the problem booklet out of the examination room.

**Problem 1**

For an integer $k \geq 2$, let $A_k$ be a $k \times k$ matrix such that its diagonal elements are 0 and all non-diagonal elements are $\dfrac{1}{k-1}$. Answer the following questions.

(1) Represent $A_3$ as $A_3 = USU^{\mathrm{T}}$ where $U$ is a $3 \times 3$ orthogonal matrix, $S$ is a $3 \times 3$ diagonal matrix $S$, and ${}^{\mathrm{T}}$ denotes the transpose.

(2) Compute the eigenvalues of $A_4$.

(3) Compute the eigenvalues of $A_k$.

(4) Compute $(A_k)^n$ for a positive integer $n$.

(5) Let $B_k$ be a matrix obtained from $A_k$ by replacing its first column vector with $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, and define $p_n$ by $p_n = (1\ 0\ \cdots\ 0)(B_k)^n \begin{pmatrix} 0 \\ \frac{1}{k-1} \\ \vdots \\ \frac{1}{k-1} \end{pmatrix}$.
Compute
$$\sum_{n=1}^{\infty} (p_n - p_{n-1})n.$$

**Problem 2**

Suppose there are given two-dimensional vectors $\boldsymbol{a}_i = \begin{pmatrix} a_{i,1} \\ a_{i,2} \end{pmatrix}$ $(i = 1, \ldots, n)$ such that they span the two-dimensional space and satisfy $\boldsymbol{a}_i^{\mathrm{T}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0$. $\boldsymbol{a}_i^{\mathrm{T}}$ denotes the transposed vector of $\boldsymbol{a}_i$. Define a region $S$ by $S = \{\boldsymbol{p} = \begin{pmatrix} x \\ y \end{pmatrix} \mid \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{p} > 0 \quad (i = 1, \ldots, n)\}$, and consider a function $f(\boldsymbol{p})$ on $S$ given by

$$f(\boldsymbol{p}) = -\sum_{i=1}^{n} \log \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{p}$$

where $\log$ is the natural logarithm. Answer the following questions.

(1) Find $\nabla f(\boldsymbol{p}) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(\boldsymbol{p}) \\ \dfrac{\partial f}{\partial y}(\boldsymbol{p}) \end{pmatrix}$ and $\nabla^2 f(\boldsymbol{p}) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(\boldsymbol{p}) & \dfrac{\partial^2 f}{\partial x \partial y}(\boldsymbol{p}) \\ \dfrac{\partial^2 f}{\partial x \partial y}(\boldsymbol{p}) & \dfrac{\partial^2 f}{\partial y^2}(\boldsymbol{p}) \end{pmatrix}$.

(2) Show that $\nabla^2 f(\boldsymbol{p})$ is positive definite at any $\boldsymbol{p} \in S$.

(3) In the case of $n = 2$, let $A$ be a $2 \times 2$ matrix whose row vectors are $\boldsymbol{a}_1^{\mathrm{T}}$ and $\boldsymbol{a}_2^{\mathrm{T}}$ in this order, and set

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \widetilde{\nabla} f(\boldsymbol{p}) = \begin{pmatrix} \dfrac{\partial f}{\partial \xi}(\boldsymbol{p}) \\ \dfrac{\partial f}{\partial \eta}(\boldsymbol{p}) \end{pmatrix}, \quad \widetilde{\nabla}^2 f(\boldsymbol{p}) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial \xi^2}(\boldsymbol{p}) & \dfrac{\partial^2 f}{\partial \xi \partial \eta}(\boldsymbol{p}) \\ \dfrac{\partial^2 f}{\partial \xi \partial \eta}(\boldsymbol{p}) & \dfrac{\partial^2 f}{\partial \eta^2}(\boldsymbol{p}) \end{pmatrix}.$$

Represent $\nabla f$, $\nabla^2 f$ by using $\widetilde{\nabla} f$, $\widetilde{\nabla}^2 f$, and $A$.

(4) In the case of $n = 2$, suppose that $\boldsymbol{p}$ on $S$ is parameterized as $\boldsymbol{p} = \boldsymbol{p}(t)$ by a parameter $t$ and it satisfies the following differential equation

$$\frac{d\boldsymbol{p}(t)}{dt} = -(\nabla^2 f(\boldsymbol{p}(t)))^{-1} \nabla f(\boldsymbol{p}(t))$$

with an initial solution $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ at $t = 0$. Find $\boldsymbol{p}(t)$ by solving this.

**Problem 3**

(1) If the probability density function $f(x)$ of continuous random variable $X$ is denoted by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0) \end{cases}$$

($\lambda$ is a non-negative constant), then we say that $X$ has an exponential distribution. Derive the mean $E(X)$ and variance $V(X)$ from the definitions of mean and variance, when $X$ is an exponential distribution.

(2) The fare of a system $Y(t)$ where $t$ denotes the time, is defined as $a$ yen for first 3 minutes and an additional $b$ yen for each additional 1 minute. (Fractional minutes are rounded up). Assuming that the usage time $g(x)$ has an exponential distribution with mean $T$, derive the mean fare $E(Y(t))$.

(3) The arrivals of users to the system defined by (2) are random. Random variable $M$ denotes the number of total usages in one day. Then divide one day into $n$ equal parts. And $Z_1, Z_2, \cdots, Z_n$ shows the number of total usages in each infinitesimal time span. Assuming that $Z_1, Z_2, \cdots, Z_n$ have the same distribution and are independent of each other, and $P\{Z_i = 1\} = \frac{\mu}{n}, P\{Z_i = 0\} = 1 - \frac{\mu}{n}$ ($\mu$ is a non-negative constant, $i = 1, 2, \cdots, n$), derive the distribution function of $P\{M = k\}$ when $n$ is sufficiently large.

(4) Derive the mean of one day's total fare of the system defined by (3).

**Problem 4**

We define the Fourier Transform of $h(x)$ in $x$ to be

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)\, e^{-i\omega x}\, dx.$$

Answer the following questions.

(1) Derive the Fourier Transform $G(\omega)$ of $g(x) = e^{-ax^2}$.

(2) For a partial differential equation of $u$ in $x$ and $t$,

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \tag{A}$$

suppose that the initial condition when $t = 0$ is $u(x, 0) = f(x)$, and the boundary conditions where $x \to \pm\infty$ are

$$\lim_{x \to \pm\infty} u(x, t) = 0.$$

Determine the ordinary differential equation for $U(\omega, t)$, Fourier Transform of $u(x, t)$, by transforming Equation (A).

(3) Derive the solution of the ordinary differential equation determined in (2). Suppose that the Fourier Transform of $f(x)$ is $F(\omega)$.

(4) Derive the solution $u(x, t)$ of Equation (A) by applying the inverse Fourier Transform to $U(\omega, t)$.

**Note** In this problem, examinees are expected to assume some unspecified boudary conditions for most possible cases observed in engineering problems

**Problem 5**

Consider the ordinary differential equation

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{A}$$

where $P(x,y,z)$, $Q(x,y,z)$, and $R(x,y,z)$ are homogeneous expressions of the same degree. Answer the following questions.

(1) Show that Equation (A) can be transformed into an equation of the following form, assuming that $x = uz$ and $y = vz$,

$$\bar{P}(u,v)du + \bar{Q}(u,v)dv + \frac{dz}{z} = 0. \tag{B}$$

(2) Prove that if Equation (A) is integrable, then Equation (B) is a total ordinary differential equation satisfying

$$\frac{\partial \bar{P}(u,v)}{\partial v} = \frac{\partial \bar{Q}(u,v)}{\partial u}.$$

(3) Make use of the above results to solve the following ordinary differential equation,

$$yzdx - z^2dy - xydz = 0.$$

**Problem 6**

Consider the integration of the complex function

$$f(z) = \frac{z^{a-1}}{1+z}, \qquad (0 < a < 1)$$

along the path that is composed of 4 parts, two circles and two lines, described in the figure. Answer the following questions.

(1) Find all the poles and the residues of $f(z)$.

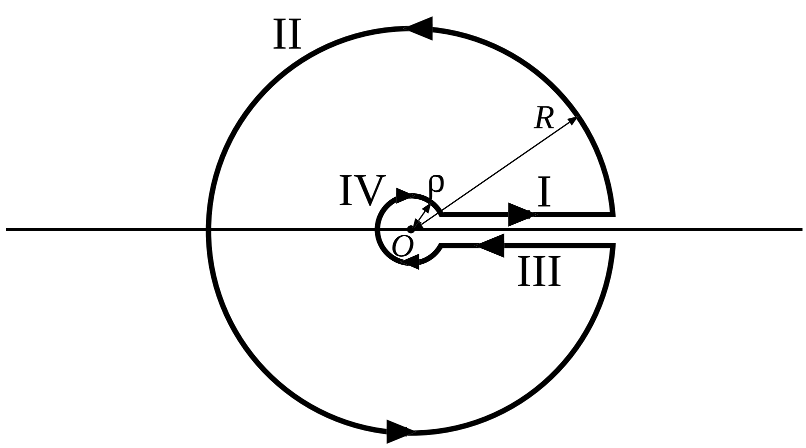
(2) Calculate the integral

$$\oint f(z)\, dz.$$

(3) Prove why the integrals along the paths II and IV shown in the diagram converge to $0$ when $R \to \infty$ and $\rho \to 0$, respectively.

(4) Calculate the following definite integral

$$\int_0^\infty \frac{x^{a-1}}{1+x}\, dx.$$




The Path of Integration