数理情報学専攻 修士課程入学試験問題

Department of Mathematical Informatics

Graduate School Entrance Examination Problem Booklet

専門科目 数理情報学

Specialized Subject: Mathematical Informatics

2023年8月21日（月）　10:00 – 13:00

August 21, 2023 (Monday)　10:00 – 13:00

5問出題，3問解答 / Answer 3 out of the 5 problems

注　意　事　項 / Instructions

(1) 試験開始の合図まで，この問題冊子を開かないこと．
Do not open this booklet until the starting signal is given.

(2) 本冊子に落丁，乱丁，印刷不鮮明の箇所などがあった場合には申し出ること．
Notify the proctor if there are missing or incorrect pages in your booklet.

(3) 本冊子には第1問から第5問まであり，日本文は4頁から13頁，英文は14頁から23頁である．5問のうち3問を日本語ないし英語で解答すること．
Five problems appear on pages 4–13 in Japanese and pages 14–23 in English in this booklet. Answer 3 problems in Japanese or English.

(4) 答案用紙3枚が渡される．1問ごとに必ず1枚の答案用紙を使用すること．止むを得ぬときは答案用紙の裏面を使用してもよい．
Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(5) 各答案用紙の指定された箇所に，受験番号およびその用紙で解答する問題番号を忘れずに記入すること．氏名は書いてはならない．
Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(6) 草稿用紙は本冊子から切り離さないこと．
Do not separate a draft sheet from the booklet.

(7) 解答に関係のない記号，符号などを記入した答案は無効とする．
Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(8) 答案用紙および問題冊子は持ち帰らないこと．
Leave the answer sheets and this booklet in the examination room.

受験番号 Examinee number	No.

上欄に受験番号を記入すること．

Fill in your examinee number.

選択した問題番号 Problem numbers			

上欄に選択した3つの問題番号を記入すること．

Fill in the three selected problem numbers.

Problem 1

For a matrix $A \in \mathbb{R}^{d\times m}$, let $a_{i,j}$ be its (i,j)-th component, $A^\top$ be its transpose, and let $\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{m} a_{i,j}^2}$. For a square matrix $A \in \mathbb{R}^{d\times d}$, its trace is $\mathrm{tr}A = \sum_{i=1}^{d} a_{i,i}$. Let I be the $d \times d$ identity matrix.

Suppose $d < m$. For given matrices $X, Y \in \mathbb{R}^{d\times m}$, let OPT($X$,$Y$) be the set of optimal solutions P of the following optimization problem:

$$\min_{P\in\mathbb{R}^{d\times d}} \|PX - Y\|_{\mathrm{F}}^2 \ \text{ subject to } \ P^\top P = I. \qquad (*)$$

Answer the following questions.

(1) Let a_j and b_j be the j-th column vectors of matrices A and B, respectively, and let $\|a_j\|_2$ be the Euclidean norm of a_j. Find a pair of matrices $X, Y \in \mathbb{R}^{d\times m}$ such that OPT(X,Y) equals the set of optimal solutions P of the following optimization problem with given matrices $A, B \in \mathbb{R}^{d\times m}$ and positive real values $w_1, \dots, w_m$:

$$\min_{P\in\mathbb{R}^{d\times d}} \sum_{j=1}^{m} w_j \|Pa_j - b_j\|_2^2 \ \text{ subject to } \ P^\top P = I.$$

(2) Show that, for matrices $X, Y \in \mathbb{R}^{d\times m}$, OPT($X$,$Y$) equals the set of optimal solutions P of the following optimization problem:

$$\min_{P\in\mathbb{R}^{d\times d}} \mathrm{tr}(PXY^\top) \ \text{ subject to } \ P^\top P = I.$$

(3) For matrices $X, Y \in \mathbb{R}^{d\times m}$, the singular value decomposition of the matrix $XY^\top$ is denoted by $XY^\top = U\Sigma V^\top$. Give an optimal solution $P \in$ OPT(X,Y) of the optimization problem $(*)$ in terms of matrices among X, Y, U, Σ, V.

Problem 2

Consider the minimization problem of a sufficiently smooth convex function $f(x)$ defined on $\mathbb{R}^d$. We assume the existence of an optimal solution x^*. We denote the gradient of $f(x)$ with respect to $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ by $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_d}(x)\right)^\top$. We denote the Euclidean norm of $x \in \mathbb{R}^d$ by $\|x\|_2$. Answer the following questions.

(1) Consider the system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t) = \dfrac{2}{t+1}(v(t) - x(t)) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}v(t) = -\dfrac{t+1}{2}\nabla f(x(t)) \end{cases} \qquad (*)$$

for $t \geq 0$. We assume that the initial conditions $x(0)$ and $v(0)$ are given. Let $E(t) = (t+1)^2(f(x(t)) - f(x^*)) + 2\|v(t) - x^*\|_2^2$. Show $\dfrac{\mathrm{d}}{\mathrm{d}t}E(t) \leq 0$ holds. Also show that there exists a constant C independent of t and $f(x(t)) - f(x^*) \leq C/t^2$ holds for every $t > 0$.

(2) As a discretization of the system of differential equations $(*)$, consider

$$\begin{cases} \delta^+ x^{(k)} = \dfrac{2hk + h + 2}{(hk+1)^2}(v^{(k+1)} - x^{(k+1)}) \\ \delta^+ v^{(k)} = -\dfrac{2hk + h + 2}{4}\nabla f(x^{(k+1)}) \end{cases} \qquad (**)$$

for $k = 0, 1, 2, \ldots$, where δ^+ is the operator that is defined for any scalar or vector sequence $\{y^{(k)}\}$ by $\delta^+ y^{(k)} = (y^{(k+1)} - y^{(k)})/h$ for some constant $h > 0$. We set $x^{(0)} = x(0)$, $v^{(0)} = v(0)$ and assume that $(**)$ has a solution. Let $E^{(k)} = (hk+1)^2(f(x^{(k)}) - f(x^*)) + 2\|v^{(k)} - x^*\|_2^2$. Show

$$\delta^+\|v^{(k)} - x^*\|_2^2 = 2(v^{(k+1)} - x^*)^\top(\delta^+ v^{(k)}) - h\|\delta^+ v^{(k)}\|_2^2,$$

and prove $\delta^+ E^{(k)} \leq 0$. You may use the identity $\delta^+(a^{(k)}b^{(k)}) = (a^{(k+1)}b^{(k+1)} - a^{(k)}b^{(k)})/h = (\delta^+ a^{(k)})b^{(k+1)} + a^{(k)}(\delta^+ b^{(k)})$ that holds for any scalar sequences $\{a^{(k)}\}, \{b^{(k)}\}$ without proving it.

(3) We assume $(**)$ has a solution. Show that there exists a constant C' independent of k and $f(x^{(k)}) - f(x^*) \leq C'/k^2$ holds for $k = 1, 2, \ldots$.

Problem 3

Let $\mathbb{R}$ be the set of all the real numbers and let $\mathbb{C}$ be the set of all the complex numbers. Let i be the imaginary unit and let e be the base of the natural logarithm. For a function $g : \mathbb{R} \to \mathbb{C}$ with period 2π, define its norm $\|g\|$ by

$$\|g\| := \sup_{\theta \in [0,2\pi]} |g(\theta)|.$$

For a positive integer n, define

$$\mathcal{T}_n := \{t : \mathbb{R} \to \mathbb{C} \mid \ t(\theta) = P(\mathrm{e}^{\mathrm{i}\theta}) \text{ holds for some polynomial } P \text{ of degree } n \text{ with complex coefficients}\}.$$

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function with period 2π, and assume that there exists $t_n \in \mathcal{T}_n$ such that

$$\|f - t_n\| \leq \frac{c}{n^3}$$

for each positive integer n. Here $c > 0$ is a constant that is independent of n. Answer the following questions.

(1) For each positive integer n, show that

$$\|t_{n+1} - t_n\| \leq \frac{2c}{n^3}$$

holds.

(2) We write the derivative of $t \in \mathcal{T}_n$ as t'. Show that the sequence of functions $\{t'_n\}$ uniformly converges on $[0, 2\pi]$ to a continuous function $s : \mathbb{R} \to \mathbb{C}$ with period 2π. You may use the following two facts without proving them.

- Let $\mathrm{C}[0, 2\pi]$ be the set of all the complex-valued continuous functions on $[0, 2\pi]$. Then, the normed space $(\mathrm{C}[0, 2\pi], \|\cdot\|)$ is complete.
- For each positive integer n and any $t \in \mathcal{T}_n$,

$$\|t'\| \leq n\|t\|$$

holds.

(3) Show that f is differentiable on $\mathbb{R}$.

Problem 4

The exponential distribution $\mathrm{Exp}(\lambda)$ for $\lambda > 0$ is the probability distribution on nonnegative real numbers with the probability density function

$$p(x) = \lambda \exp(-\lambda x).$$

Let $X_1, X_2, \ldots$ be a sequence of independent random variables obeying $\mathrm{Exp}(\lambda)$. Answer the following questions.

(1) For $c > 0$, find the conditional probability density function of $X_1 - c$ given $X_1 \geq c$.

(2) Let $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. Find the probability density functions of Y and $Z - Y$.

(3) Find the probability density function of $X_1 + \cdots + X_n$.

(4) For $a > 0$, let N be the random variable defined by $X_1 + \cdots + X_N \leq a < X_1 + \cdots + X_{N+1}$. Let $N = 0$ if $a < X_1$. Find the maximum likelihood estimate $\widehat{\lambda}(N)$ of λ based on N. Also, find the mean and the variance of $\widehat{\lambda}(N)$.

Problem 5

Let $\mathbb{N}$ be the set of natural numbers (positive integers). Consider the following problem of assigning jobs to machines. For natural numbers n and m, let $W = \{w_1, w_2, \ldots, w_n\}$ denote the set of jobs and $V = \{v_1, v_2, \ldots, v_m\}$ denote the set of machines. For each job $w_i \in W$, let $\ell_{w_i} \in \mathbb{N}$ be the processing time for job w_i and $V_{w_i} \subseteq V$ the set of machines that can process job w_i. A mapping $\pi : W \to V$ is called a *feasible assignment* if

$$\pi(w_i) \in V_{w_i}$$

for each $w_i \in W$. Given a feasible assignment π, we define the *load* $T(\pi, v_j)$ of machine $v_j \in V$ as

$$T(\pi, v_j) = \sum_{w_i \in \pi^{-1}(v_j)} \ell_{w_i},$$

and the *makespan* $T(\pi)$ under π as

$$T(\pi) = \max_{v_j \in V} T(\pi, v_j).$$

The *minimum makespan problem* is the problem of finding a feasible assignment π that minimizes the makespan $T(\pi)$ over all feasible assignments. Answer the following questions.

(1) Suppose that all the processing times ℓ_{w_i} of jobs are identical, i.e., $\ell_{w_i} = \ell_{w_j}$ for every pair of jobs $w_i, w_j \in W$. Provide a polynomial-time algorithm for the minimum makespan problem.

(2) Consider the following greedy algorithm.

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(Initialization:) Let π be a mapping π : ∅ → V over the empty set.
for i = 1, 2, ..., n do
    Choose an arbitrary machine v* from argmin_{v_j ∈ V_{w_i}} T(π, v_j).
    Extend π to a mapping over {w_1, ..., w_i} by π(w_i) = v*.
end for
Output π.
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Suppose that $V_{w_i} = V$ for every $w_i \in W$. Let π_{ALG} be the output of the greedy algorithm and T^* be the optimal makespan. Show that $T(\pi_{\mathrm{ALG}}) \leq 2T^*$.