数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成２５年８月２０日（火）　１０：００～１３：００

５問出題，３問解答

This booklet is an informal English translation of the original examination booklet. Answer in Japanese or English.

Answer three out of the five problems.

Please note:

(1) Do not open this booklet until the starting signal is given.

(2) Notify the supervisor if there are missing or incorrect pages in your booklet.

(3) Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(4) Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(5) Do not separate the draft paper from this booklet.

(6) Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(7) Leave the answer sheets and this booklet in the examination room.

Examinee number	No.

Fill in your examinee number.

Problem numbers			

Fill in numbers of the three answered problems.

Problem 1

Let X be a real symmetric tridiagonal matrix of order n defined by

$$X = \begin{bmatrix} a_1 & -b_1 & 0 & \cdots & 0 \\ -b_1 & a_2 & -b_2 & \ddots & \vdots \\ 0 & -b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & -b_{n-1} \\ 0 & \cdots & 0 & -b_{n-1} & a_n \end{bmatrix},$$

where $b_j \neq 0$ $(j = 1, 2, \ldots, n-1)$.

(1) For a real number μ, let $p_k(\mu)$ denote the leading principal minor of order k $(k = 1, 2, \ldots, n)$, i.e., the determinant of the submatrix indexed by the first k rows and columns, of matrix $X - \mu I$, where I is the identity matrix. Show that $p_{k-1}(\mu) = 0$ implies $p_k(\mu)p_{k-2}(\mu) < 0$, where $k = 3, \ldots, n$.

(2) Let λ be an eigenvalue of X. Obtain an explicit form of the corresponding eigenvector $\boldsymbol{u}$ whose first component u_1 is normalized to one, using some of a_i, b_j, and $p_k(\lambda)$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n-1;\ k = 1, 2, \ldots, n)$.

Problem 2

The matrix exponential for a complex square matrix A is defined by

$$\mathrm{e}^{A} := I + \sum_{k=1}^{\infty} \frac{A^k}{k!},$$

where I is the identity matrix.

(1) Obtain e^{A} for $A = \begin{bmatrix} 0 & a \\ 0 & 2\pi \mathrm{i} \end{bmatrix}$, where a is a real constant and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

(2) Let A and B be complex square matrices. Show that $\mathrm{e}^{t(A+B)} = \mathrm{e}^{tA}\mathrm{e}^{tB}$ holds for any real number t if and only if $AB = BA$.

(3) Find a pair of complex square matrices A and B such that both $\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}$ and $AB \neq BA$ hold.

(4) Let A be a real square matrix of order n. Consider the ordinary differential equation

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = A\boldsymbol{y}(t) \qquad (*)$$

for a function $\boldsymbol{y} : \mathbb{R} \to \mathbb{R}^n$, where $\mathbb{R}$ denotes the set of real numbers. Show that the solution with the initial condition $\boldsymbol{y}(0) = \boldsymbol{v}$ is given by $\boldsymbol{y}(t) = \mathrm{e}^{tA}\boldsymbol{v}$.

(5) Suppose that the coefficient matrix A in $(*)$ is expressed as $A = P + Q$ with real matrices P and Q. We now consider approximating $\boldsymbol{y}$ by the solution $\tilde{\boldsymbol{y}}$ of the time discretization

$$\begin{aligned} \tilde{\boldsymbol{y}}((k+1)h) &= \mathrm{e}^{hP}\mathrm{e}^{hQ}\tilde{\boldsymbol{y}}(kh) \qquad (k = 0, 1, 2, \ldots), \\ \tilde{\boldsymbol{y}}(0) &= \boldsymbol{y}(0) \end{aligned}$$

with $h > 0$. We assume that e^{hP} and e^{hQ} can be computed exactly. Discuss the accuracy of this approximation at $t = h$ by evaluating the error $\|\tilde{\boldsymbol{y}}(h) - \boldsymbol{y}(h)\|$, where the symbol $\|\cdot\|$ designates a norm.

Problem 3

Let N be the number of fish in a pond. To estimate N, we first catch n $(\leq N)$ fish, mark them, and release them into the pond. After a sufficiently long time, we catch fish one by one until m $(\leq n)$ marked fish are collected. Let X be the number of unmarked fish caught at this time. Here, N, n, and m are at least one.

(1) Show that the equation

$$\binom{N}{n} = \sum_{k=m}^{N-n+m} \binom{k-1}{m-1} \binom{N-k}{n-m}$$

holds.

(2) For a nonnegative integer x, obtain the probability $p(x; N, n, m)$ that $X = x$.

(3) Calculate the expected value $\mathrm{E}[X]$.

(4) Construct an unbiased estimator of N.

Problem 4

Consider a population composed of N (≥ 2) species. Denote by $x_i(t)$ the fraction of species i ($i = 1, 2, \ldots, N$) in the population at time t (≥ 0). Note that $\sum_{i=1}^{N} x_i(t) = 1$ is satisfied for any $t \geq 0$. Each $x_i(t)$ obeys the differential equation

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = x_i(t) \left(\sum_{j=1}^{N} A_{ij} x_j(t) - \sum_{k=1}^{N} \sum_{j=1}^{N} A_{kj} x_k(t) x_j(t) \right) \qquad (i = 1, 2, \ldots, N), \qquad (*)$$

where each A_{ij} ($i = 1, 2, \ldots, N; j = 1, 2, \ldots, N$) is a real constant.

(1) Suppose that there is an equilibrium $(p_1, p_2, \ldots, p_N)$ in which all the species coexist. Show that the value of $\sum_{j=1}^{N} A_{ij} p_j$ is independent of i in such an equilibrium.

Note: We say that $(p_1, p_2, \ldots, p_N)$ is an equilibrium of equation $(*)$ when $x_i(t) = p_i$ ($i = 1, 2, \ldots, N$) satisfy equation $(*)$. We say that all the species coexist in the equilibrium $(p_1, p_2, \ldots, p_N)$ when $p_i > 0$ holds true for all i.

(2) Take arbitrary real constants c_j ($j = 1, 2, \ldots, N$), and consider replacing each A_{ij} by $A_{ij} + c_j$. Show that equation $(*)$ is invariant under this replacement (i.e., show that the differential equation obtained as a result of the replacement is equivalent to equation $(*)$).

In the following, we set $N = 2$ and define $z(t) := x_1(t)$.

(3) Obtain the differential equation satisfied by $z(t)$ when $A_{11} = A_{22} = 0$, $A_{12} = a$, and $A_{21} = b$. Also obtain the condition on a and b for $\lim_{t \to \infty} z(t) = 1$ to hold true irrespectively of the initial value $z(0)$ ($0 < z(0) < 1$).

(4) For general A_{11}, A_{12}, A_{21}, and A_{22} values, obtain the condition on A_{11}, A_{12}, A_{21}, and A_{22} for $\lim_{t \to \infty} z(t) = 1$ to hold true irrespectively of the initial value $z(0)$ ($0 < z(0) < 1$).

(5) Obtain the condition on A_{11}, A_{12}, A_{21}, and A_{22} for the coexistence of the two species to be realized in the limit $t \to \infty$ irrespectively of the initial value $z(0)$ ($0 < z(0) < 1$).

Problem 5

For a positive integer n (≥ 2) and integers a, b, c, and d, let $L(a, b, c, d, n)$ denote the set of all pairs (x, y) of integers x and y that satisfy congruence equations

$$ax + by \equiv c \pmod{n}, \qquad bx - (a + b)y \equiv d \pmod{n}.$$

(1) Obtain all the elements (x, y) of $L(5, 3, 0, 0, 21)$ that lie in the range of $0 \leq x \leq 20$ and $0 \leq y \leq 20$, and plot these points (x, y) on the xy-plane.

(2) Determine whether $L(5, 3, 3, 5, 21)$ is empty or not.

(3) Show that, if $(u, v) \in L(a, b, b, a, n)$, then

$$\begin{bmatrix} u & v \\ -v & u - v \end{bmatrix} \begin{bmatrix} a & b \\ b & -a - b \end{bmatrix} \equiv \begin{bmatrix} b & a \\ a & -a - b \end{bmatrix} \pmod{n}.$$

Here, the congruence for two matrices means the congruence for each of the corresponding entries of the matrices.

(4) Show that, if $L(a, b, b, a, n)$ is not empty, then $L(a, b, 0, 0, n) = L(b, a, 0, 0, n)$ holds.