# 数理情報学専攻

# 修士課程入学試験問題

# 専門科目 数理情報学

平成２４年８月２１日（火）　１０：００〜１３：００

**５問出題，３問解答**

This booklet is an informal English translation of the original examination booklet. Answer in Japanese or English.

**Answer three out of the five problems.**

Please note:

(1) Do not open this booklet until the starting signal is given.

(2) Notify the supervisor if there are missing or incorrect pages in your booklet.

(3) Three answer sheets will be given. Use one sheet per problem. If necessary, you may use the back of the sheet.

(4) Fill in the examinee number and the problem number in the designated place of each answer sheet. Do not put your name.

(5) Do not separate the draft paper from this booklet.

(6) Any answer sheet with marks or symbols unrelated to the answer will be invalid.

(7) Leave the answer sheets and this booklet in the examination room.

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| Examinee number | No. |

Fill in your examinee number.

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| Problem numbers | | | |

Fill in numbers of the three answered problems.

**Problem 1**

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers with $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$, and define a square matrix $A = (a_{ij})$ of order $n$ by

$$a_{ij} = \min\{\alpha_i, \alpha_j\}.$$

For $n = 3$, for example, we have

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_3 & \alpha_3 \end{bmatrix}.$$

(1) Obtain the determinant of $A$ for a general $n$.

(2) Show that $A$ is positive definite for a general $n$.

(3) Obtain the inverse of $A$ for $n = 4$.

(4) Obtain the inverse of $A$ for a general $n$.

**Problem 2**

Consider an opinion formation model on a connected undirected graph $G$ with $N\,(\geq 2)$ vertices defined as follows. We assume that each vertex has either black or white opinion. Until all the vertices possess the same opinion, we iterate the following steps (a), (b), and (c) in this order. We set the "strength" of the black and white opinions to $r$ $(0 < r \neq 1)$ and 1, respectively.

(a) Select a vertex $v$ in $G$ with the probability proportional to the strength of the opinion at each vertex.

(b) Select a vertex $v'$ which is adjacent to $v$ with the equal probability.

(c) Convert the opinion at $v'$ to that at $v$.

(1) Assume that $G$ is a complete graph (see Fig. 2.1 for the case of $N = 6$). Denote by $x_i$ $(0 \leq i \leq N)$ the probability that all the vertices eventually have the black opinion when exactly $i$ vertices have the black opinion as an initial condition. Obtain the recurrence equation satisfied by $x_{i-1}, x_i$, and $x_{i+1}$ $(1 \leq i \leq N-1)$.

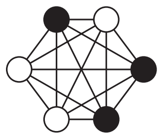


Fig. 2.1.

(2) Obtain $x_i$ $(0 \leq i \leq N)$ defined in (1).

(3) Assume that $G$ is a star, in which one vertex called the hub is adjacent to all the other $N-1$ vertices and the $N-1$ non-hub vertices are adjacent only to the hub (see Fig. 2.2 for the case of $N = 6$). Let $N \geq 3$ and assume that we start with an initial condition such that two vertices have the black opinion and $N-2$ vertices have the white opinion. Which two vertices must initially have the black opinion to maximize the probability that all the vertices eventually have the black opinion?

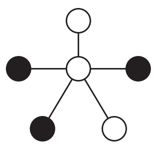


Fig. 2.2.

**Problem 3**

Let $f(z)$ be a holomorphic function on a region in the complex plane and let $u(z)$ and $v(z)$ denote the real and imaginary parts of $f(z)$, respectively: $f(z) = u(z) + \mathrm{i}\, v(z)$. Let $x$ and $y$ denote the real and imaginary parts of $z$, respectively. Consider $u(z)$ and $v(z)$ as functions of $(x, y)$ and write

$$u(z) = u(x + \mathrm{i}\, y) = u(x, y), \quad v(z) = v(x + \mathrm{i}\, y) = v(x, y).$$

(1) Write down the Cauchy-Riemann equations that $u(x, y)$ and $v(x, y)$ satisfy.

(2) Does there exist a holomorphic function $f(z)$ whose real part is identical to the following $u(x, y)$? If so, find the function $f(z)$.

(a) $u(x, y) = x^2 - y^2$

(b) $u(x, y) = x^2 + y^2$

(c) $u(x, y) = \dfrac{x}{x^2 + y^2}$ (We assume that $(x, y) \neq (0, 0)$.)

(3) Suppose that a function $f(z)$ is holomorphic in a region containing the real axis and the upper half-plane, and converges to 0 uniformly when $0 \leq \arg z \leq \pi$ and $|z| \to \infty$. Then prove the following for $y > 0$:

$$\begin{aligned} f(z) &= \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\xi - x}{(\xi - x)^2 + y^2} f(\xi) \mathrm{d}\xi \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(\xi - x)^2 + y^2} f(\xi) \mathrm{d}\xi. \end{aligned}$$

(Hint) Consider the integral of $f(\xi)\left(\dfrac{1}{\xi - z} \pm \dfrac{1}{\xi - \overline{z}}\right)$ along the contour $C$ in Fig. 3.1.

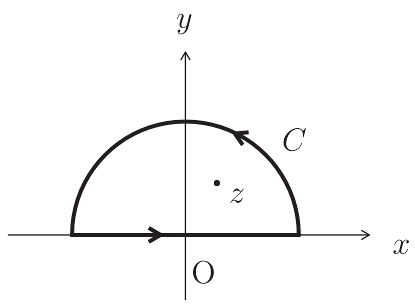



Fig. 3.1. Contour $C$

(4) Under the conditions given in (3), represent $f(z)$ by $u(\xi, 0)$ (the values of $u$ on the real axis).

**Problem 4**

In engineering and physics, it is often observed that nonlinear systems driven by external periodic forcing exhibit oscillatory phenomena. As such an example, let us consider the following second-order ordinary differential equation with respect to a real-valued function $x(t)$ of time $t$:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + (1-\epsilon)x(t) + \epsilon\, x(t)^3 = \epsilon A \cos t, \qquad \cdots \quad (*1)$$

where $A$ and $\epsilon$ are positive constants.

(1) Define $y(t)$ as $y(t) = \dfrac{\mathrm{d}x}{\mathrm{d}t}(t)$. By applying the transformation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \cdots \quad (*2)$$

to Equation $(*1)$, a first-order simultaneous differential equation with respect to functions $u(t)$ and $v(t)$ can be derived. The derived equation can be expressed in the following form:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t}(t) &= \epsilon F(u(t), v(t), t), \\ \frac{\mathrm{d}v}{\mathrm{d}t}(t) &= \epsilon G(u(t), v(t), t), \end{aligned}$$

where $F(u, v, t)$ and $G(u, v, t)$ are functions having period $2\pi$ with respect to $t$. Obtain the functions $F(u, v, t)$ and $G(u, v, t)$.

(2) Obtain the following functions of $(u, v)$ defined in terms of $F(u, v, t)$ and $G(u, v, t)$ in (1) above:

$$\begin{aligned} \tilde{F}(u, v) &= \frac{1}{2\pi} \int_0^{2\pi} F(u, v, \tau)\mathrm{d}\tau, \\ \tilde{G}(u, v) &= \frac{1}{2\pi} \int_0^{2\pi} G(u, v, \tau)\mathrm{d}\tau. \end{aligned}$$

You may use the following formulas: $\displaystyle\int_0^{2\pi} \sin^2\tau \mathrm{d}\tau = \int_0^{2\pi} \cos^2\tau \mathrm{d}\tau = \pi$, $\displaystyle\int_0^{2\pi} \sin\tau\cos\tau \mathrm{d}\tau = 0$, $\displaystyle\int_0^{2\pi} \sin^4\tau \mathrm{d}\tau = \int_0^{2\pi} \cos^4\tau \mathrm{d}\tau = \frac{3\pi}{4}$, $\displaystyle\int_0^{2\pi} \sin^3\tau\cos\tau \mathrm{d}\tau = \int_0^{2\pi} \sin\tau\cos^3\tau \mathrm{d}\tau = 0$, $\displaystyle\int_0^{2\pi} \sin^2\tau\cos^2\tau \mathrm{d}\tau = \frac{\pi}{4}$.

(3) By applying the inverse transformation of Equation $(*2)$ to a constant solution $(\tilde{u}(t), \tilde{v}(t)) = (u_0, v_0)$ of the differential equation

$$\begin{aligned} \frac{\mathrm{d}\tilde{u}}{\mathrm{d}t}(t) &= \epsilon\, \tilde{F}(\tilde{u}(t), \tilde{v}(t)), \\ \frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}(t) &= \epsilon\, \tilde{G}(\tilde{u}(t), \tilde{v}(t)), \end{aligned}$$

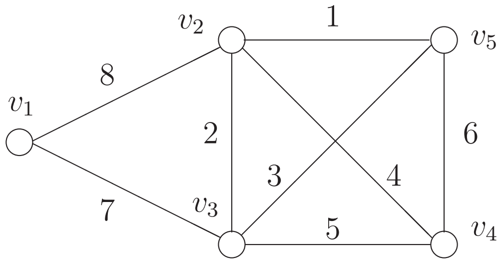
a periodic solution $\tilde{x}(t)$ can be obtained. Let us call such a periodic solution an approximate periodic solution[†] for Equation $(*1)$. When $A = \dfrac{1}{4}$, obtain all the approximate periodic solutions $\tilde{x}(t)$ for Equation $(*1)$.

[†] While this solution does not satisfy Equation $(*1)$ exactly, it is used, for sufficiently small $\epsilon$, as an approximate solution to investigate oscillatory phenomena (the averaging method).

**Problem 5**

Let $G = (V, E)$ denote a connected undirected graph, and let $w_e$ $(e \in E)$ be positive edge weights. We consider the problem of computing a spanning tree that minimizes the sum of the edge weights, where a tree is called spanning if it connects all the vertices in $V$. This is a fundamental problem in network design. For this problem, we consider the following greedy algorithm, where $m = |E|$.

Greedy Algorithm
**Step 1.** $T := \emptyset$;
**Step 2.** Arrange the edges $e_1, e_2, \ldots, e_m$ so that $w_{e_1} \le w_{e_2} \le \cdots \le w_{e_m}$.
**Step 3.** **for** $i = 1, 2, \ldots, m$ **do**
  **if** $T \cup \{e_i\}$ contains no cycle **then** $T := T \cup \{e_i\}$;
 **end**
**Step 4.** Output $T$ and halt.




($V = \{v_1, v_2, v_3, v_4, v_5\}$; The edge weight is attached to each edge.)

Fig. 5.1. An example of graph $G$ with edge weights.

(1) Apply the greedy algorithm to the example in Fig. 5.1.

(2) Let $T\,(\subseteq E)$ be a spanning tree and $e$ be an edge with $e \notin T$. Then $T \cup \{e\}$ contains exactly one cycle, which is denoted by $C(T, e)$. Show that, for a spanning tree $T$, the following statements (a) and (b) are equivalent.

(a) $T$ is a spanning tree with the minimum $\displaystyle\sum_{e \in T} w_e$.

(b) For any edge $e \notin T$ and any edge $e' \in C(T, e)$, we have $w_e \ge w_{e'}$.

(3) The greedy algorithm computes a spanning tree $T$ with the minimum $\displaystyle\sum_{e \in T} w_e$. Prove this by using the result of (2).

(4) The spanning tree $T$ obtained by the greedy algorithm also minimizes the maximum edge weight $\displaystyle\max_{e \in T} w_e$. Prove this.