数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成２０年８月２６日（火）　１０：００～１３：００

５問出題，３問解答

- This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 $\sim$ Problem 5.**
- **Answer in Japanese or English.**

Problem 1

For an $n \times n$ complex matrix $A = [a_{ij}]_{i,j=1,2,\dots,n}$, we set $r_i = \sum_{j \neq i, 1 \leq j \leq n} |a_{ij}| \quad (i = 1, 2, \dots, n)$. The following is a well-known theorem for the nonsingularity of matrix A.

> Theorem I. If
> $$|a_{ii}| > r_i \qquad (i = 1, 2, \dots, n), \qquad (*)$$
> then matrix A is nonsingular.

A proof of the theorem is as follows:

(Proof) Suppose, to the contrary, that A satisfies $(*)$ and is singular. Then there is an $\boldsymbol{x} \neq \boldsymbol{0}$ such that

$$A\boldsymbol{x} = \boldsymbol{0}, \quad \text{i.e.,} \quad \sum_{j=1}^{n} a_{ij} x_j = 0 \quad (i = 1, 2, \dots, n)\,.$$

Set $|x_t| = \max_{1 \leq j \leq n} |x_j|$. For this t, we see $\sum_{j=1}^{n} a_{tj} x_j = 0$, or equivalently,

$$a_{tt} x_t = - \sum_{j \neq t, 1 \leq j \leq n} a_{tj} x_j.$$

From this equality we see

$$|a_{tt}||x_t| \leq \sum_{j \neq t, 1 \leq j \leq n} |a_{tj}||x_j| \leq \Big(\sum_{j \neq t, 1 \leq j \leq n} |a_{tj}| \Big) |x_t| = r_t |x_t|.$$

By dividing both sides by $|x_t| \, (\neq 0)$, we obtain $|a_{tt}| \leq r_t$, which is a contradiction to $(*)$.

(1) With the idea of the proof above, prove the following theorem.

> Theorem II. Let $n \geq 2$. If
> $$|a_{ii}||a_{jj}| > r_i r_j \qquad (i \neq j;\ i, j = 1, 2, \dots, n), \qquad (**)$$
> then matrix A is nonsingular.

[Hint] In addition to x_t, consider the component x_s of $\boldsymbol{x}$ whose absolute value is the second largest.

(2) Assume $n \geq 2$. Show that Theorem II is stronger than Theorem I by showing the following (i) and (ii).

(i) If a matrix A satisfies $(*)$, then it satisfies $(**)$.

(ii) There exists a matrix A that satisfies $(**)$, but not $(*)$.

Problem 2

For a complex number z, we denote its real part by $\mathrm{Re}(z)$, and we define the sign function by

$$\mathrm{sign}(z) = \begin{cases} 1, & \mathrm{Re}(z) > 0 \\ -1, & \mathrm{Re}(z) < 0 \\ 0, & \mathrm{Re}(z) = 0. \end{cases}$$

(1) Give the Newton iteration formula for finding a root of $z^2 - 1 = 0$, denoting the initial value by z_0 and the iterated solutions by $z_1, z_2, \ldots$. Furthermore, show that this iteration never breaks down (that is, the denominator of the iteration formula never becomes zero) for any initial value z_0 satisfying $\mathrm{Re}(z_0) \neq 0$.

(2) With the iterated solutions z_k obtained in the procedure above, define a sequence f_k $(k = 0, 1, 2, \ldots)$ of functions of z_0 by

$$f_k(z_0) = \frac{z_k - \mathrm{sign}(z_0)}{z_k + \mathrm{sign}(z_0)},$$

where we assume that $\mathrm{Re}(z_0) \neq 0$. Show that the denominator of $f_k(z_0)$ never becomes zero and that

$$f_{k+1}(z_0) = \big(f_k(z_0)\big)^2 \qquad (k = 0, 1, 2, \ldots).$$

(3) Utilizing the results above, show that the Newton iteration defined in (1) always converges for any initial value z_0 satisfying $\mathrm{Re}(z_0) \neq 0$ and that

$$\lim_{k \to \infty} z_k = \mathrm{sign}(z_0).$$

(4) For a given $n \times n$ complex matrix Z_0, consider the iteration:

$$Z_{k+1} = \frac{Z_k + {Z_k}^{-1}}{2} \qquad (k = 0, 1, 2, \ldots).$$

Show that, if the matrix Z_0 is diagonalizable$^{(\star)}$ and has no eigenvalue on the imaginary axis, the iteration never breaks down and Z_k is convergent.

($\star$) We say that Z_0 is diagonalizable when there exists a nonsingular matrix S such that the matrix $S^{-1} Z_0 S$ is diagonal.

Problem 3

(1) Suppose that an $n \times n$ real matrix $Q = [q_{ij}]_{i,j=1,2,\ldots,n}$ is an orthogonal matrix with its first row given by $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$. We put

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = Q\boldsymbol{x} = \begin{bmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ q_{21} & \cdots & q_{2n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ is an n-dimensional real vector. Show $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=2}^{n} {y_i}^2$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

(2) Suppose that random variables $x_1, x_2, \ldots, x_n$ are independently distributed according to the normal distribution with mean μ and variance σ^2, i.e., the simultaneous probability density function is given by

$$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right].$$

Obtain the simultaneous probability density function of $y_1, y_2, \ldots, y_n$.

(3) Suppose that random variables $x_1, x_2, \ldots, x_n$ are independently distributed according to the normal distribution with mean μ and variance σ^2. Consider an estimator

$$\hat{\sigma}^2 = f(n)\sum_{i=1}^{n}(x_i - \bar{x})^2$$

of the variance σ^2, where $f(n)$ is a function of n. Obtain $f(n)$ that makes the estimator $\hat{\sigma}^2$ unbiased.

(4) Suppose that a random variable x is distributed according to the normal distribution with mean μ and variance σ^2. Obtain the fourth moment $\mathrm{E}[(x - \mu)^4]$ about the mean.

(5) Suppose that random variables $x_1, x_2, \ldots, x_n$ are independently distributed according to the normal distribution with mean μ and variance σ^2. Obtain the function $g(n)$ of n that minimizes the mean squared error $\mathrm{E}[(\hat{\sigma}^2 - \sigma^2)^2]$ of the estimator

$$\hat{\sigma}^2 = g(n)\sum_{i=1}^{n}(x_i - \bar{x})^2$$

of the variance σ^2.

Problem 4

The logistic map f, widely used in the study of nonlinear dynamics and chaos, is a one-dimensional map from the closed interval $I = [0, 1]$ to I defined by

$$f(x) = ax(1 - x) \qquad (x \in I),$$

where a is a parameter satisfying $0 < a \leq 4$. Let $x_0 \in I$ be an initial value and define $x_{n+1} = f(x_n)$ $(n = 0, 1, 2, \dots)$.

In the following we assume $a = 4$ and consider the distribution of $x_0, x_1, x_2, \dots, x_{N-1}$ on I. It is known that, for almost all x_0, this distribution has an identical limit distribution as $N \to \infty$, the limit distribution is described by a density function $p(x)$ continuous on the open interval $(0, 1)$, and

$$\int_{[0,x]} p(u)\mathrm{d}u = \int_{f^{-1}([0,x])} p(u)\mathrm{d}u \qquad (x \in I) \tag{$*$}$$

holds. Here, $f^{-1}([0, x]) = \{u \in I : f(u) \in [0, x]\}$ and $\int_I p(x)\mathrm{d}x = 1$. Let us find this density function $p(x)$.

(1) Define maps h and g from I to I by

$$\begin{aligned} h(x) &= \frac{2}{\pi} \arcsin \sqrt{x} \qquad (x \in I), \\ g(y) &= 1 - 2\left| y - \frac{1}{2} \right| \qquad (y \in I). \end{aligned}$$

Show that $g\big(h(x)\big) = h\big(f(x)\big)$ for all $x \in I$. Furthermore, show that the change of variables $y_n = h(x_n)$ $(n = 0, 1, 2, \dots)$ transforms $x_{n+1} = f(x_n)$ to $y_{n+1} = g(y_n)$. Here g is called tent map.

(2) Show that the density function $q(y) = 1$ $(y \in I)$ of the uniform distribution on I satisfies

$$\int_{[0,y]} q(v)\mathrm{d}v = \int_{g^{-1}([0,y])} q(v)\mathrm{d}v \qquad (y \in I), \tag{$**$}$$

where g is the tent map and $g^{-1}([0, y]) = \{v \in I : g(v) \in [0, y]\}$.

(3) Derive the expression that is obtained from $(**)$ by the change of variables $y = h(x)$ and $v = h(u)$. By comparing that expression with $(*)$ and by using the result of (2), find $p(x)$, where no proof of uniqueness is required.

Problem 5

Let $G = (V, E)$ denote a directed graph with a finite vertex set V and an edge set $E\,(\subseteq V \times V)$. Each edge $(u, v) \in E$ is associated with a real number $\ell(u, v)$ representing its length.

A path P in G is a sequence of edges which is of the form of $(v^{(0)}, v^{(1)}), (v^{(1)}, v^{(2)}), \ldots,$ $(v^{(k-1)}, v^{(k)})$. P is called a path from u to v if $u = v^{(0)}$ and $v = v^{(k)}$ hold in addition. Here a path may contain repeated edges and/or vertices. The length of path P is defined as the sum of the edge lengths in it.

We consider the following procedure, ShortestPath, to compute a shortest path (i.e., a path of minimum length) from a given start vertex $s \in V$ to every vertex $v \in V$. Here we assume that there exists a path from s to every vertex $v \in V$ in G. In the procedure, value $d(v)$ $(v \in V)$ is either a real number or "$+\infty$," and value $p(v)$ $(v \in V)$ is either a vertex in V or "**nil**" (a symbol which does not belong to V).

Procedure ShortestPath

Step 1. **for** each $v \in V$ **do**
  $p(v) :=$ **nil**
  if $v = s$ **then** $d(v) := 0$
  else $d(v) := +\infty$;

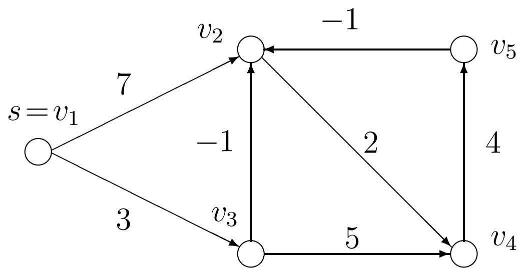
Step 2. **while** (the first iteration) or
  (d was not updated during the previous iteration) **do**
  for each $(u, v) \in E$ **do**
   if $d(u) < +\infty$ and $d(v) > d(u) + \ell(u, v)$
   then $d(v) := d(u) + \ell(u, v)$, $p(v) := u$.

(1) Apply Procedure ShortestPath to Example 1 in Figure 1. Here in the **for**-loop in Step 2, edge (u, v) is scanned in the following order: (v_5, v_2), (v_4, v_5), (v_2, v_4), (v_3, v_4), (v_3, v_2), (v_1, v_2), (v_1, v_3).

(1-1) Check if the procedure halts or not. If it halts, then show a shortest path from s to every vertex.

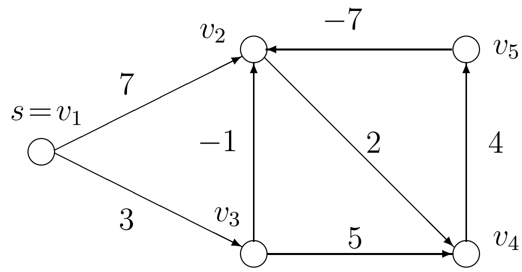
(1-2) Show how the values of d and p are updated during the iterations in Step 2. If the procedure halts, then show the values of d and p until the procedure halts; Otherwise, show the values at the end of each iteration, from the first to the 6th (make a table like the one below).

	$d(v_1)$	$d(v_2)$	$d(v_3)$	$d(v_4)$	$d(v_5)$	$p(v_1)$	$p(v_2)$	$p(v_3)$	$p(v_4)$	$p(v_5)$
Before the 1st iteration	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	**nil**	**nil**	**nil**	**nil**	**nil**
End of the 1st iteration										
End of the 2nd iteration										
End of the 3rd iteration										
⋮										



(edge length is attached to each edge)

Fig. 1. Example 1 of a graph G with an edge length ℓ.



(edge length is attached to each edge)

Fig. 2. Example 2 of a graph G with an edge length ℓ.

(2) Apply Procedure ShortestPath to Example 2 in Figure 2. Here in the **for**-loop in Step 2, edge (u, v) is scanned in the same order as in (1): (v_5, v_2), (v_4, v_5), (v_2, v_4), (v_3, v_4), (v_3, v_2), (v_1, v_2), (v_1, v_3).

(2-1) Check if the procedure halts or not. If it halts, then show a shortest path from s to every vertex.

(2-2) Show how the values of d and p are updated during the iterations in Step 2. If the procedure halts, then show the values of d and p until the procedure halts; Otherwise, show the values at the end of each iteration, from the first to the 6th.

(3) From the results of (1) and (2), we see that Procedure ShortestPath does not always halt. Explain when the procedure does not halt by referring to "cycle" (i.e., a path whose start vertex and end vertex are the same) in graph G.

(4) Consider a system of linear inequalities

$$x_v - x_u \le \ell(u, v) \qquad ((u, v) \in E)$$

in $|V|$ variables x_v $(v \in V)$. Give a necessary and sufficient condition, with a proof, for this system of linear inequalities to have a feasible solution, by referring to procedure ShortestPath. Here the result of (3) may be used in the proof.