数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成１９年８月２１日（火）　１０：００～１３：００

５問出題，３問解答

◦ This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 ∼ Problem 5.**
- **Answer in Japanese or English.**

Problem 1

Let A be an $n \times n$ matrix, and b be an n-dimensional vector. For a natural number k, let $W_k = W_k(A,b)$ be the subspace spanned by $\{b, Ab, A^2b, \ldots, A^{k-1}b\}$ and denote the dimension of W_k by d_k. Here, the components of a matrix or a vector are complex numbers.

(1) Prove that $W_k \subseteq W_{k+1}$.

(2) Prove that $W_k = W_{k+1}$ implies $W_{k+1} = W_{k+2}$.

(3) Prove the inequality

$$d_{k+1} - d_k \geq d_{k+2} - d_{k+1} \qquad (k = 1, 2, \ldots).$$

(4) Prove that $W_k(A - \lambda I, b) = W_k(A, b)$ holds for any complex number λ, where I denotes the $n \times n$ unit matrix.

(5) Consider the case where $n = 3$ and

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find a necessary and sufficient condition for $d_3 = 3$ in terms of α, b_1, b_2, and b_3.

(6) Consider the case where $n = 5$ and

$$A = \begin{bmatrix} \alpha & 1 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}.$$

Find a necessary and sufficient condition for $d_5 = 5$ in terms of α, β, b_1, b_2, b_3, b_4, and b_5.

Problem 2

Let us consider a predator-prey system consisting of three species: P, Q, and R, where P preys on Q, Q on R, and R on P. As a mathematical model describing the time evolution of the number of each species in such a predator-prey system, the following differential equation (cyclic Lotka–Volterra equation) is known.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p(t) \\ q(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} p(t)(q(t)-r(t)) \\ q(t)(r(t)-p(t)) \\ r(t)(p(t)-q(t)) \end{pmatrix}. \qquad (*)$$

The real numbers $p(t)$, $q(t)$, and $r(t)$, respectively, denote the proportion of the number of each species P, Q, and R, at time $t \geq 0$.

Let us consider the solution of the system $(*)$, assuming that a set of initial values at $t = 0$ is given. The initial values $p(0)$, $q(0)$, and $r(0)$ are assumed to be positive, and satisfy $p(0) + q(0) + r(0) = 1$.

(1) Show that $p(t) + q(t) + r(t) = 1$ for any $t > 0$. Show also that, for any $t > 0$, the value $I(p(t), q(t), r(t)) = p(t)q(t)r(t)$ remains constant, and $0 < p(t) < 1$, $0 < q(t) < 1$, and $0 < r(t) < 1$ hold.

(2) The proportion $p(t)$ increases when $(p(t), q(t))$ belongs to a certain region. Show the increasing region on the two-dimensional (p, q)-plane (take $p(t)$ as the horizontal axis, and $q(t)$ the vertical axis). Similarly, show the region where $q(t)$ increases.

(3) Find the stationary solution (fixed point) of the differential equation $(*)$. If the initial values do not lie on the fixed point, $(p(t), q(t))$ moves periodically along a closed curve $\{(p, q) : I(p, q, 1-p-q) = \text{const.}\}$. Sketch the outline of the closed curve, and show by arrows the direction in which $(p(t), q(t))$ moves as time passes.

Next, in order to solve the problem approximately, let us discretize the differential equation $(*)$ by the Euler method to obtain

$$\frac{1}{h}\begin{pmatrix} p_{n+1}-p_n \\ q_{n+1}-q_n \\ r_{n+1}-r_n \end{pmatrix} = \begin{pmatrix} p_n(q_n-r_n) \\ q_n(r_n-p_n) \\ r_n(p_n-q_n) \end{pmatrix},$$

where $h > 0$ is the time mesh size. The values p_n, q_n, and r_n $(n = 0, 1, 2, \ldots)$, respectively, denote the approximations of $p(nh)$, $q(nh)$, and $r(nh)$, with $p_0 = p(0)$, $q_0 = q(0)$, and $r_0 = r(0)$.

(4) Prove that, if the time mesh size h is chosen small enough, for all initial values and for all $n = 1, 2, \ldots$, it holds that $0 < p_n < 1$, $0 < q_n < 1$, and $0 < r_n < 1$.

(5) Suppose that the initial values do not lie on the stationary solution of $(*)$. Prove that, if the time mesh size h is chosen small enough, $I_n = I(p_n, q_n, r_n)$ is strictly monotonically decreasing with respect to n. Sketch on the (p, q)-plane the outline of the trajectory of the approximate solution $\{(p_n, q_n) : n = 0,\ 1,\ 2,\ \ldots\}$.

Problem 3

Consider a discrete-time Markov process that has a countable state set S, which we denote by $\{X(n) : n = 0, 1, 2, \ldots\}$. Assume that the state-transition probability does not depend on time. Let $Q^{(n)}(j,k)$ be the probability of going from $j \in S$ to $k \in S$ in n steps, that is,

$$Q^{(n)}(j,k) = \mathrm{P}(X(n) = k \mid X(0) = j).$$

Let $f^{(n)}(j,k)$ be the probability that the Markov process starts from $X(0) = j$ and arrives at k in n (≥ 1) steps for the first time. We define $f^{(0)}(j,k) = 0$ $(\forall j, \forall k)$. Furthermore, we define the generating functions of $Q^{(n)}(j,k)$ and $f^{(n)}(j,k)$ respectively by

$$\begin{aligned} \overline{Q}_{jk}(z) &= \sum_{n=0}^{\infty} Q^{(n)}(j,k) z^n, \\ \overline{f}_{jk}(z) &= \sum_{n=0}^{\infty} f^{(n)}(j,k) z^n. \end{aligned}$$

(1) Prove

$$\overline{Q}_{jk}(z) = \delta_{jk} + \overline{f}_{jk}(z)\overline{Q}_{kk}(z) \quad (\forall j, \forall k \in S),$$

where δ_{jk} is equal to 1 when $j = k$ and 0 otherwise.

(2) The time for the Markov process starting from j to return to j for the first time is denoted by

$$T_j = \min\{n \geq 1 : X(n) = j, X(0) = j\},$$

where $T_j = \infty$ if such an n does not exist. State j is defined to be recurrent if T_j is finite with probability 1. When j is recurrent, find the value of $\overline{f}_{jj}(1)$.

(3) Prove that the condition

$$\sum_{n=0}^{\infty} Q^{(n)}(j,j) = \infty$$

is necessary and sufficient for state j to be recurrent.

(4) As the Markov process $\{X(n) : n = 0, 1, 2, \ldots\}$, let us consider a one-dimensional random walk defined by

$$S = \{0, \pm 1, \pm 2, \ldots\}, \quad Q^{(1)}(j,k) = \begin{cases} p & (k = j+1) \\ 1-p & (k = j-1) \\ 0 & (\text{otherwise}) \end{cases}.$$

Find $\overline{Q}_{00}(z)$, $\overline{Q}_{10}(z)$, and $\overline{f}_{10}(z)$. Use the following relation if necessary:

$$\frac{1}{\sqrt{1+t}} = \sum_{i=0}^{\infty} \frac{(2i)!}{(i!)^2} \left(-\frac{t}{4}\right)^i.$$

(5) Consider a random walk on vertices of the infinite graph in which each vertex has d adjacent vertices ($d \geq 3$) and there is a unique path between any two vertices (tree graph). The position of the random walk moves to each neighbor of the current vertex with probability $1/d$, independently of the time. Let $X(n)$ be the number of edges between the starting vertex at time 0 and the current vertex at time n. Then $\{X(n) : n = 0, 1, \ldots\}$ is a Markov process with state set $S = \{0, 1, 2, \ldots\}$. For example, $X(1) = 1$ always holds. From $n = 1$ to the time when the starting vertex is reached for the first time, $\{X(n) : n = 0, 1, \ldots\}$ can be regarded as a random walk of the type given in (4) for an appropriate value of p. Express such p in terms of d.

(6) For $\{X(n) : n = 0, 1, \ldots\}$ given in (5), find $\overline{f}_{00}(z)$ and show that the state $0 \in S$ is not recurrent.

Problem 4

Let O be a view point, and P and Q be two other points. The angle formed by the two half lines OP and OQ emanating from O is called the *visual angle* of P and Q at O. At time t_1 of a clear night, at some place in Japan, both a bright star A and a satellite B were seen in the same direction as a mountain peak P, and the visual angle of P and the North Star was α. After time interval t_2, on the same night, i.e., at time $t_1 + t_2$, the visual angle of the star A and the peak P was β, and the satellite B, which circled almost once around the earth, was seen in the same direction as P. Then, the satellite B circled around the earth once more and at time $t_1 + 2t_2$ it was seen again in the same direction as the mountain peak P. Suppose that the earth rotates exactly once in one day, that the North Star is on the axis of the rotation of the earth, and that the satellite B moves along a great circle fixed with respect to the center of the earth at a constant speed. Let θ denote the angle between the normal vector to the plane containing the great circle and the axis of the rotation of the earth and let ω denote the angular velocity of the satellite B with respect to the center of the earth.

(1) Find the time interval t_2.

(2) Concerning the movement of the satellite B along the great circle, obtain feasible values of (θ, ω).

Problem 5

Let $G = (V^+ \cup V^-, E)$ be a bipartite graph with a finite vertex set $V^+ \cup V^-$ (with $V^+ \cap V^- = \emptyset$) and an edge set $E \subseteq \{\{u, v\} : u \in V^+, v \in V^-\}$. An edge subset $M \subseteq E$ is called a matching of G if edges in M share no end vertex. Namely, $M = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}\} \subseteq E$ is a matching of G if $\{u_i, v_i\} \cap \{u_j, v_j\} = \emptyset$ holds for any i, j $(i \neq j)$. This problem considers an algorithm to compute a matching of G with the maximum size, called a maximum matching.

(1) Give an example that the following simple greedy algorithm cannot compute a maximum matching in general. Here $E = \{e_1, e_2, \ldots, e_m\}$.

Step 1. $M := \emptyset$.

Step 2. **for** $i = 1$ **to** m **do**

if $M \cup \{e_i\}$ is a matching **then** $M := M \cup \{e_i\}$.

Step 3. output M and halt.

(2) Let M_1 and M_2 be two matchings of G. Show properties of the subgraph $G' = (V^+ \cup V^-, M_1 \Delta M_2)$ of G whose edge set is the symmetrical difference of M_1 and M_2, i.e., $M_1 \Delta M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$.

(3) Let M_1 be a matching of G, and let M_2 be a maximum matching of G. Then, by using G' in (2), show a necessary and sufficient condition that M_1 is maximum.

(4) Let $\vec{G}$ be a directed graph obtained from G and a matching M_1 by orienting edges in M_1 from V^- to V^+ and edges in $E \setminus M_1$ from V^+ to V^-. By using $\vec{G}$, show a necessary and sufficient condition that M_1 is maximum. (Make use of the result of (3).)

(5) By using the result of (4), construct an algorithm to compute a maximum matching and analyze its time complexity.