数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成１８年８月２２日（火）　１０：００～１３：００

５問出題，３問解答

◦ This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 ∼ Problem 5.**
- **Answer in Japanese or English.**

Problem 1

Let $A = (a_{ij})$ be a real symmetric matrix of order n. Denote by λ the maximum eigenvalue of A, and put $S = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$. We consider an inequality

$$\lambda \geq \frac{1}{n} S. \qquad (*)$$

Answer the following questions (1)–(3).

(1) Prove that

$$\lambda = \frac{1}{n} S$$

holds if the vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, with all elements equal to one, is an eigenvector for the eigenvalue λ.

(2) Prove the inequality $(*)$ when $n = 2$.

(3) Does the inequality $(*)$ hold for general n? If so, give a proof. Otherwise, show a counterexample.

Problem 2

(1) Suppose that X is a random variable that takes values $x_1, x_2, \ldots, x_K$ and Y is a random variable that takes values $y_1, y_2, \ldots, y_L$.

(1-1) Express the probabilities $\mathrm{P}(Y = y_j)$ in terms of the probabilities $\mathrm{P}(X = x_i)$ and conditional probabilities $\mathrm{P}(Y = y_j | X = x_i)$, where $i = 1, 2, \ldots, K$, and $j = 1, 2, \ldots, L$.

(1-2) Show that the expectation $\mathrm{E}[Y]$ is equal to $\mathrm{E}[\mathrm{E}[Y|X]]$, where $\mathrm{E}[Y|X]$ is the conditional expectation. Express the variance $\mathrm{V}[Y]$ in terms of the conditional variance $\mathrm{V}[Y|X]$ and $\mathrm{E}[Y|X]$.

(2) Suppose that there are three coins A, B, and C. The probabilities of heads of the coins A, B, and C are p, q, and r, respectively. Consider the following experiment:

The first step**:** Toss coin C.

The second step**:** If coin C shows head at the first step, coin A is tossed N times. If coin C shows tail at the first step, coin B is tossed N times.

Let Y be the number of heads at the second step. Find the expectation $\mathrm{E}[Y]$ and variance $\mathrm{V}[Y]$.

(3) On non-rainy days, customers arrive at a shop according to a Poisson process with arrival rate λ_0. On rainy days, customers arrive at the shop according to a Poisson process with arrival rate λ_1. Assume that $\lambda_0 > \lambda_1 > 0$. Let r be the probability of a rainy day.

Let Y be the number of customer arrivals per unit time. Find the expectation $\mathrm{E}[Y]$ and variance $\mathrm{V}[Y]$. Obtain r that maximizes $\mathrm{V}[Y]$.

Problem 3

Let

$$\varphi(z) = \mathrm{e}^{-z^2}\,.$$

Answer the following questions (1)–(3).

(1) Let $s(\neq 0)$ be a real number. By evaluating the integral of $\varphi(z)$ along the contour C given in Fig. 1 and making $R \to \infty$, obtain the Fourier transform of $\varphi(x)$:

$$\widehat{\varphi}(s) = \int_{-\infty}^{\infty} \varphi(x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x\,.$$

Here you may use $\widehat{\varphi}(0) = \displaystyle\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = \sqrt{\pi}\,.$

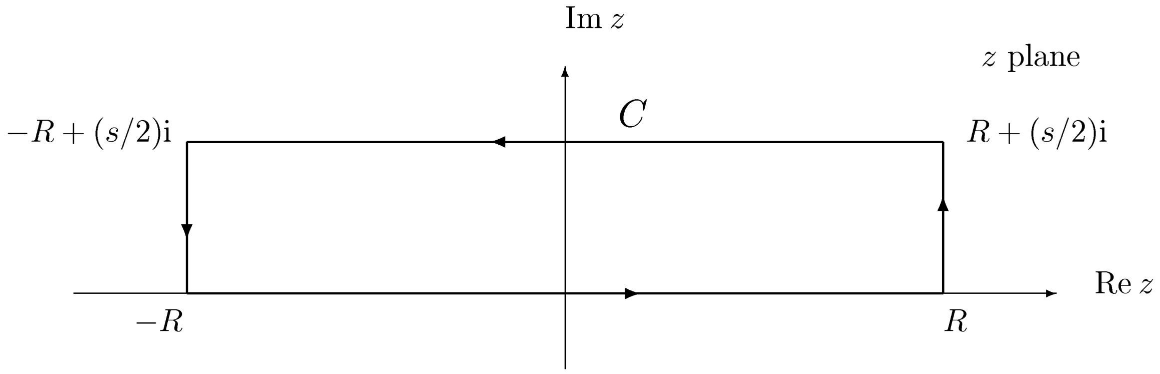

Fig. 1. The contour C

(2) For a positive real number h, consider a series

$$F(x) = \sum_{m=-\infty}^{\infty} \varphi(x+mh)\,.$$

The series is absolutely convergent, and represents a periodic function of x with the period h. Using an identity $\mathrm{e}^{-\mathrm{i}\,(2\pi n/h)\,(x+mh)} = \mathrm{e}^{-\mathrm{i}\,(2\pi n/h)\,x}$ $(m, n$: integers$)$, obtain the Fourier coefficients of the periodic function $F(x)$:

$$c_n = \frac{1}{h}\int_0^h F(x)\,\mathrm{e}^{-\mathrm{i}\,(2\pi n/h)\,x}\,\mathrm{d}x\,.$$

(3) Consider an approximation of the integral $\displaystyle\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x$ by the rectangular rule $h\displaystyle\sum_{m=-\infty}^{\infty} \varphi(mh)$. Show that the approximation error is estimated as

$$h\sum_{m=-\infty}^{\infty} \varphi(mh) - \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x \risingdotseq 2\sqrt{\pi}\,\mathrm{e}^{-\pi^2/h^2}$$

for a sufficiently small h.

Problem 4

A truss is a structure consisting of m independent members, or bars, connected by pin-joints, or nodes. For simplicity, we restrict ourselves to the behavior of trusses in the two-dimensional space, and assume the small deformation. Fig. 1 shows an example of a truss consisting of 10 members. In the following, we consider an optimization problem to find a truss design that is strongest in the sense of an appropriate measure.

Letting a_i denote the cross-sectional area of the i th member, the vector $\boldsymbol{a} \in \mathbb{R}^m$ consisting of $a_1, \dots, a_m$ is considered as the design variables vector. Let $\boldsymbol{u} \in \mathbb{R}^n$ denote the vector of nodal displacements, where n denotes the number of degrees of freedom of the truss. The stored energy, or the strain energy, per unit volume of the ith member is given by

$$\pi_i(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{b}_i^\top \boldsymbol{u})^2,$$

where $\boldsymbol{b}_i \in \mathbb{R}^n$ is a constant vector. Let l_i denote the length of the i th member (Note that the assumption of small deformation permits us to regard l_i as constant). Then the energy stored in the i th member can be written as $a_i l_i \pi_i(\boldsymbol{u})$.

Let $\boldsymbol{f} \in \mathbb{R}^n$ denote the vector of external forces applied at nodes of the truss. For a given $\boldsymbol{f}$, the total potential energy $\Phi(\boldsymbol{a}, \boldsymbol{u})$ is defined as

$$\Phi(\boldsymbol{a}, \boldsymbol{u}) = \sum_{i=1}^{m} a_i l_i \pi_i(\boldsymbol{u}) - \boldsymbol{f}^\top \boldsymbol{u}.$$

Note that $\Phi(\boldsymbol{a}, \boldsymbol{u})$ attains the minimum with respect to $\boldsymbol{u}$ at the equilibrium state.

Recall the 10-bar truss illustrated in Fig. 1. We may define the displacement vector $\boldsymbol{u} \in \mathbb{R}^8$ by assigning each component according to the arrows shown in Fig. 1. By using the same system of coordinates, the external forces illustrated in Fig. 2 can be expressed as $\boldsymbol{f} = (0, -100, 0, -150, 0, 0, 0, 0)^\top$. The corresponding equilibrium state is shown in Fig. 2 by solid lines.

For a given external load $\boldsymbol{f} (\neq \boldsymbol{0})$, the function w defined by

$$w(\boldsymbol{a}) = -2 \min_{\boldsymbol{u} \in \mathbb{R}^n} \Phi(\boldsymbol{a}, \boldsymbol{u})$$

is referred to as the compliance. It can be shown that the compliance $w(\boldsymbol{a})$ coincides with twice the energy stored in a truss at the equilibrium state. Hence, in structural engineering, it is usually desired to find a design variables vector that has the smallest possible compliance for a given $\boldsymbol{f} (\neq \boldsymbol{0})$. This motivates us to consider the following minimization problem of the compliance:

$$\begin{aligned} \min_{\boldsymbol{a} \in \mathbb{R}^m} \quad & w(\boldsymbol{a}) \\ \text{s.t.} \quad & \boldsymbol{l}^\top \boldsymbol{a} \leq V_0, \quad \boldsymbol{a} \geq \boldsymbol{0}, \end{aligned}$$

where $V_0 > 0$ denotes a given upper bound of the total volume of members. Fig. 3 illustrates an optimal design which minimizes the compliance under the external load $\boldsymbol{f}$ defined by Fig. 2.

Answer the following questions (1)–(3).

(1) Show that the compliance $w(\boldsymbol{a})$ is a convex function, i. e., that the inequality

$$w(\lambda \boldsymbol{a}_1 + (1-\lambda)\boldsymbol{a}_2) \leq \lambda w(\boldsymbol{a}_1) + (1-\lambda) w(\boldsymbol{a}_2) \qquad (0 \leq \lambda \leq 1)$$

holds for any $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^m$.

(2) For a given $\boldsymbol{a}$, it is known that the displacement vector $\overline{\boldsymbol{u}}(\boldsymbol{a})$ at the equilibrium state coincides with $\boldsymbol{u}$ which attains

$$\min_{\boldsymbol{u} \in \mathbb{R}^n} \Phi(\boldsymbol{a}, \boldsymbol{u}).$$

Show that the sensitivity coefficient of compliance, denoted by $\dfrac{\partial w}{\partial a_i}$, with respect to the cross-sectional area of the i th member can be written as

$$\frac{\partial w}{\partial a_i} = -l_i (\boldsymbol{b}_i^\top \overline{\boldsymbol{u}}(\boldsymbol{a}))^2.$$

(3) Let $\boldsymbol{a}^* = (a_i^*)$ denote an optimal solution of the minimization problem of the compliance. By deriving the Karush–Kuhn–Tucker conditions satisfied at an optimal solution of this optimization problem, show the fact that all members satisfying $a_i^* > 0$ share the same value of strain energy $\pi_i(\overline{\boldsymbol{u}}(\boldsymbol{a}^*))$.

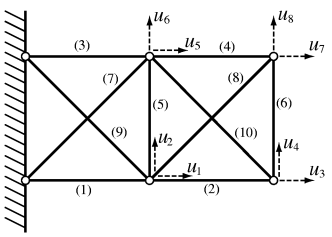

Fig. 1. A truss ($m = 10$, $n = 8$)

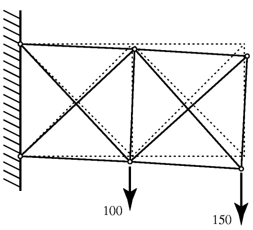

Fig. 2. An external load $\boldsymbol{f}$ and the corresponding equilibrium state $\overline{\boldsymbol{u}}(\boldsymbol{a})$.

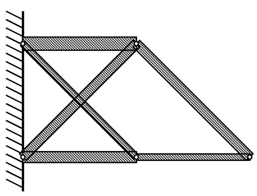

Fig. 3. An optimal solution of the minimization problem of the compliance (Note that some members have been removed that have vanishing cross-sectional areas at the optimal solution).

Problem 5

In the x-y plane, there are M different circles $C_1, C_2, \dots, C_M$ and N different lines $L_1, L_2, \dots, L_N$. We denote by K_{ij} the number of intersection points between circle C_i and line L_j, and let $K = \sum_{i=1}^{M} \sum_{j=1}^{N} K_{ij}$. For instance, we have $K = 7$, 15 and 10 for the examples in Figs. 1, 2, and 3, respectively. We consider the problem of computing K, i.e., the number of intersection points between the circles and the lines. Here circle C_i is represented by its center (s_i, t_i) and radius $r_i\,(> 0)$, and line L_j is represented by the coefficients a_j, b_j and d_j of its equation $a_j x + b_j y = d_j$.

It is clear that the problem can be solved in $\mathrm{O}(MN)$ time in a straightforward manner, i.e., first count intersection points to obtain K_{ij} for each pair of circle C_i and line L_j, and then sum them up.

In the following, you are asked to design more efficient algorithms for the problem. Answer the following questions (1)–(3).

(1) Consider the case of M concentric circles centered at the origin $(0, 0)$ and N lines parallel to the y-axis (as in Fig. 1), where the radii r_i of circles C_i satisfy $r_1 < r_2 < \cdots < r_M$, and the x-coordinates $x_j\,(= d_j/a_j)$ of lines L_j satisfy $|x_1| \le |x_2| \le \cdots \le |x_N|$.

(1-1) Let $P_{\alpha,\beta}$ be the number of intersection points between circles $C_\alpha, C_{\alpha+1}, \dots, C_M$ and lines $L_\beta, L_{\beta+1}, \dots, L_N$, i.e., $P_{\alpha,\beta} = \sum_{i=\alpha}^{M} \sum_{j=\beta}^{N} K_{ij}$. Express $P_{\alpha,\beta}$ by $P_{\alpha+1,\beta}$ and $P_{\alpha,\beta+1}$, based on the relation between r_α and x_β.

(1-2) Based on (1-1), design a linear-time algorithm for computing the number K of intersection points between the circles and the lines from $r_1, r_2, \dots, r_M$ and $x_1, x_2, \dots, x_N$.

(2) Consider the case of concentric circles centered at (s, t) and arbitrary lines (as in Fig. 2). Design an efficient algorithm for computing the number K of intersection points between the circles and the lines, and evaluate the time complexity of your algorithm.

(3) Consider the case of arbitrary circles and the lines that are parallel to the y-axis (as in Fig. 3). Let x_j denote the x-coordinate of line L_j (i.e., $x_j = d_j/a_j$), and assume that no pair of circle C_i and line L_j exists such that L_j is tangent to C_i. To compute the number of intersection points, we may start by sorting the x-coordinates of both leftmost and rightmost points of all the circles:

$$s_i - r_i,\ s_i + r_i \ \ (i = 1, 2, \dots, M),$$

and the x-coordinates of all the lines:

$$x_j \ \ (j = 1, 2, \dots, N),$$

to get $z_1 \leq z_2 \leq \ldots \leq z_{2M+N}$, and then compute the number K of intersection points by traversing $z_1, z_2 \ldots, z_{2M+N}$. Design an efficient algorithm in this way, and evaluate the time complexity of your algorithm.

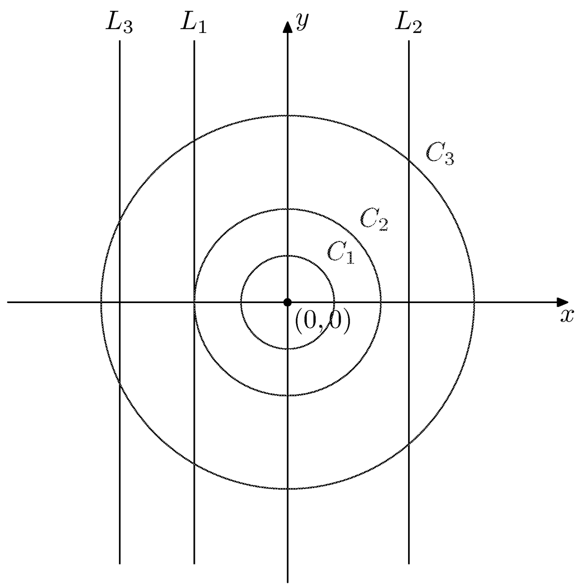

Fig. 1. Concentric circles and parallel lines

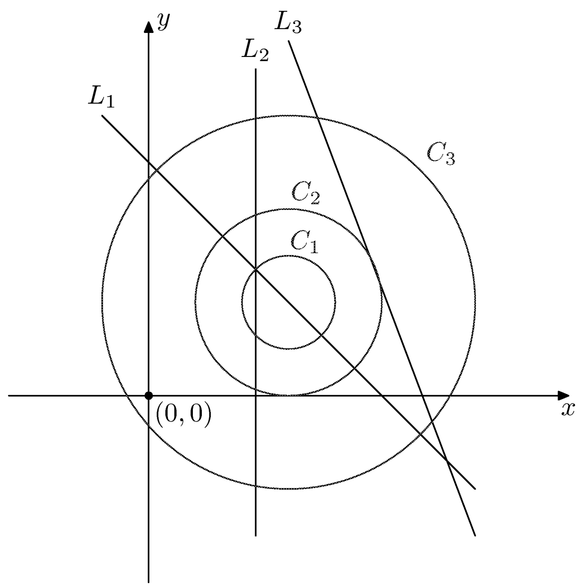

Fig. 2. Concentric circles and arbitrary lines

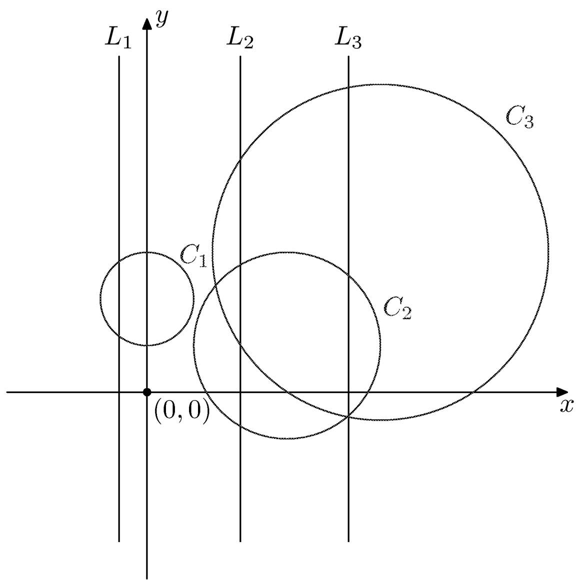

Fig. 3. Arbitrary circles and parallel lines