数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成１７年８月２３日（火）　１０：００～１３：００

５問出題，３問解答

○ This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 ～ Problem 5.**
- **Answer in Japanese or English.**

Problem 1

Let C_n be an $n \times n$ square matrix defined by

$$C_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

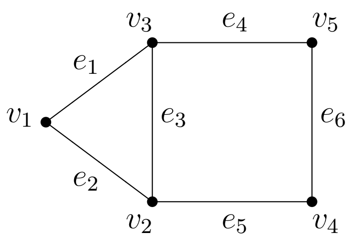
Answer the following questions (1)–(4).

(1) Show that if n is even, then the inverse matrix of C_n does not exist.

(2) Show that if n is odd, then the set of all the column vectors of C_n is linearly independent.

(3) Show that if the inverse matrix ${C_n}^{-1}$ of C_n exists, then $2{C_n}^{-1}$ is an integer matrix.

(4) Let G and $M(G)$ be an undirected graph and its incidence matrix, respectively, depicted below:



An undirected graph G.

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	1	1	0	1	0
v_3	1	0	1	1	0	0
v_4	0	0	0	0	1	1
v_5	0	0	0	1	0	1

The incidence matrix $M(G)$.

Let H be the real linear subspace spanned by the column vectors $e_1, e_2, \cdots, e_6$ of $M(G)$. Obtain every edge subset of G such that the corresponding set of column vectors of $M(G)$ is a basis of H.

Problem 2

Answer the following questions (1)–(3).

(1) Suppose that a straight line $y = \beta_1 x + \beta_2$ is fitted to data composed of pairs of real numbers (x_i, y_i) $(i = 1, \ldots, n)$ and that estimators $\hat{\beta}_1$ of β_1 and $\hat{\beta}_2$ of β_2 are obtained by the least squares method. Let

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad X = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}, \quad \text{and} \quad \hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}.$$

The rank of X is assumed to be 2. Find $\hat{\boldsymbol{\beta}}$. Find the $n \times n$ matrix P determined by X satisfying

$$X\hat{\boldsymbol{\beta}} = P\mathbf{y}$$

and show that the equalities $P^2 = P$ and $P^\top = P$ hold, where $P^\top$ designates the transpose of P.

(2) Suppose that a quadratic curve $y = \gamma_1 x^2 + \gamma_2 x + \gamma_3$ is fitted to the data considered in (1) composed of pairs of real numbers (x_i, y_i) $(i = 1, \ldots, n)$ and estimators $\hat{\gamma}_1$ of γ_1, $\hat{\gamma}_2$ of γ_2, and $\hat{\gamma}_3$ of γ_3 are obtained by the least squares method. Let

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad Z = \begin{pmatrix} {x_1}^2 & x_1 & 1 \\ {x_2}^2 & x_2 & 1 \\ \vdots & \vdots & \\ {x_n}^2 & x_n & 1 \end{pmatrix}, \quad \text{and} \quad \hat{\boldsymbol{\gamma}} = \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \\ \hat{\gamma}_3 \end{pmatrix}.$$

The rank of Z is assumed to be 3. Find the $n \times n$ matrix Q determined by Z satisfying

$$Z\hat{\boldsymbol{\gamma}} = Q\mathbf{y}.$$

Show the equalities

$$QP = PQ = P,$$

where P is the matrix obtained in (1).

(3) We assume a model in which each y_i is independently distributed according to the normal distribution with mean $b_1 x_i + b_2$ and variance σ^2 for the data considered in (1) composed of pairs of real numbers (x_i, y_i) $(i = 1, \ldots, n)$. Let $\hat{b}_1$, $\hat{b}_2$, and $\hat{\sigma}$ be the maximum likelihood estimators of the unknown parameters b_1, b_2, and σ, respectively. Show the equalities

$$\hat{b}_1 = \hat{\beta}_1 \quad \text{and} \quad \hat{b}_2 = \hat{\beta}_2,$$

where $\hat{\beta}_1$ and $\hat{\beta}_2$ are defined in (1).

Problem 3

Two different kinds of crystals start growing from $\mathrm{P}_1 = (0, 0)$ and $\mathrm{P}_2 = (a, 0)$ $(a > 0)$ in the xy-plane simultaneously. The crystal from P_1 grows at speed 1, and the crystal from P_2 grows at speed k $(k > 1)$; both of the crystals grow uniformly in any direction. A crystal cannot enter the region where the other crystal has already grown. Therefore, when a crystal meets the other, the growth in that direction ends; the faster crystal reaches behind the other by growing around the other crystal region. Let $(r(\theta), \theta)$ be the polar-coordinate representation of the common boundary curve of the two crystal regions obtained after a sufficiently long time. That is, $r(\theta)$ denotes the distance from the origin to the point on the boundary curve in the direction θ measured from the positive x direction counterclockwise.

Fig. 1 shows an example of the situation where $k = 2$. Thin curves in the figure represent the boundaries of the regions obtained by several units of time, and the thick curve represents the boundary of the two crystal regions obtained after a sufficiently long time.

Answer the following questions (1)–(4).

(1) Find the equation satisfied by the curve $(r(\theta), \theta)$ in the area $x \geq 0$.

(2) Give the integral expression that represents the length of a general curve $(r(\theta), \theta)$ associated with the range $\theta_1 \leq \theta \leq \theta_2$.

(3) Find the equation satisfied by $r(\theta)$ in the area $x \leq 0$ and $y \geq 0$.

(4) Find constants α and β such that $r(\theta) = \alpha \mathrm{e}^{\beta\theta}$ satisfies the equation obtained in (3).

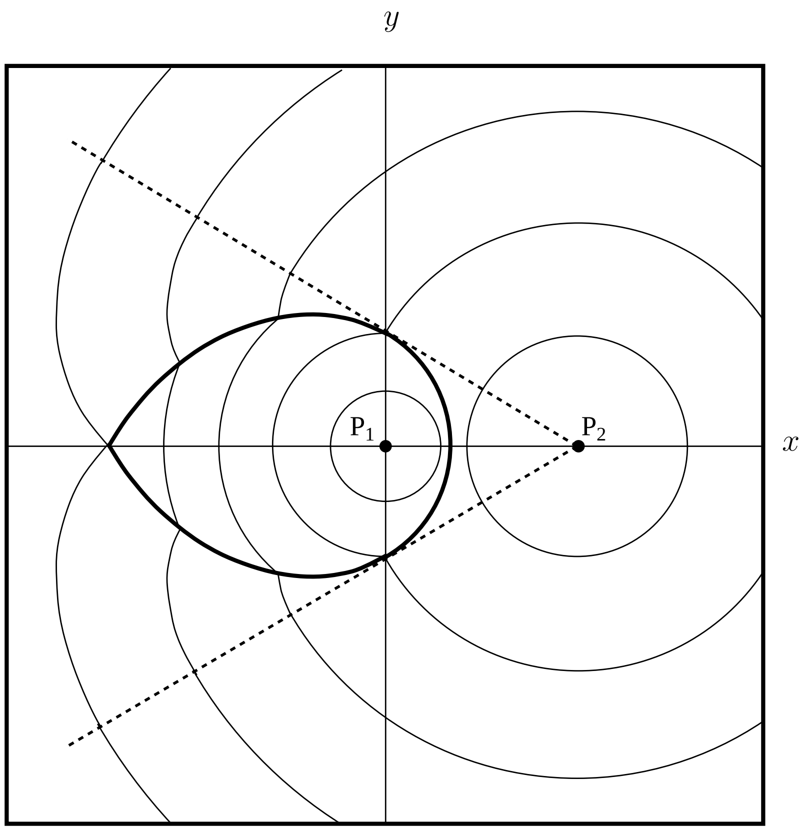


Fig. 1. The two growing crystals ($k = 2$).

Problem 4

A biological neuron and its firing are often described by a mathematical model composed of a linear electrical circuit and a discharge device. In order to construct such a model, let us first consider a linear electrical circuit with a capacitor C, a resistor R, and a current source $I(t)$ in Fig. 2, where t represents time. Dynamics of the voltage $V(t)$ across the capacitance, which is measured by the potential at point a relative to that at point b, is described by the following differential equation:

$$C\frac{\mathrm{d}V(t)}{\mathrm{d}t} + \frac{V(t)}{R} = I(t).$$

Next, let us consider a circuit in Fig. 3, which is obtained by connecting a discharge device to the linear electrical circuit of Fig. 2. The behavior of the discharge model of Fig. 3 is as follows: $V(t)$ changes in response to $I(t)$ for $t \geq 0$ from the initial condition $V(0) = 0$ at $t = 0$. When $V(t)$ reaches the threshold voltage V_{th} at $t = T$, where $V_{\mathrm{th}} > 0$, the discharge device operates and $V(t)$ is instantaneously reset to $V(T) = 0$ by discharge. After this instantaneous discharge, $V(t)$ restarts to change in response to $I(t)$ and if $V(t)$ reaches V_{th} again, it is instantaneously reset to 0 with discharge. The process will repeat for $t \geq 0$. The discharge corresponds to firing of a biological neuron.

Answer the following questions (1)–(4) on the circuits in Figs. 2 and 3. In all questions, consider $V(t)$ for $t \geq 0$ with the initial condition $V(0) = 0$.

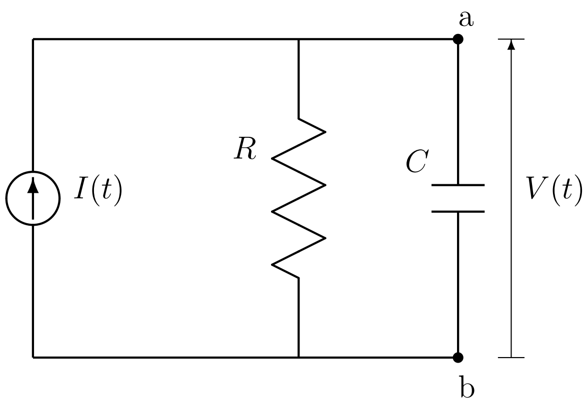


Fig. 2. A linear electrical circuit.

(1) In the circuit of Fig. 2, when $I(t)$ is a step current, namely $I(t) = 0$ for $t < 0$ and $I(t) = I_0$ for $t \geq 0$, where I_0 is a positive parameter, find $V(t)$. Next, when the same step current $I(t)$ is applied to the circuit of Fig. 3, find the condition which I_0 must satisfy so that discharge never occurs. Furthermore, when the

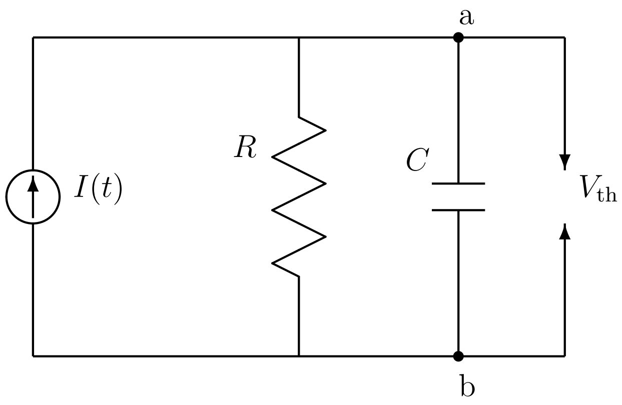


Fig. 3. A discharge model.

above condition is not satisfied, in other words when discharge occurs, find the first discharge time T as a function of I_0 and illustrate the function on the plane with the horizontal axis I_0 and the vertical axis T.

(2) Find $V(t)$ when an impulse current $I(t) = \delta(t - T_0)$ is applied to the circuit of Fig. 2, where $T_0 > 0$ and $\delta(t)$ is Dirac's delta function such that $\delta(t) = 0$ for $t \neq 0$ and

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1.$$

(3) Let us consider the following periodic train of impulse currents

$$I(t) = \beta \sum_{j=1}^{\infty} \delta(t - jT_0),$$

where β is a positive parameter. When the above periodic train $I(t)$ is applied to the circuit of Fig. 2, illustrate the waveform of $V(t)$ in the steady state after a sufficiently long time and find the maximum and the minimum values of $V(t)$. Next, when the same periodic train $I(t)$ is applied to the circuit of Fig. 3, find the condition which β must satisfy so that discharge never occurs. Furthermore, when the above condition is not satisfied, find the first discharge time T.

(4) Suppose that the current source $I(t)$ changes according to the following differential equation:

$$\frac{\mathrm{d}^2 I(t)}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}I(t)}{\mathrm{d}t} + \gamma^2 I(t) = 0$$

with the initial conditions $I(0) = 0$ and $\frac{\mathrm{d}I}{\mathrm{d}t}(0) = K$, where γ and K are positive parameters. Solve this equation and illustrate the waveform of $I(t)$. This current $I(t)$ is an example of α-functions that are often used to represent inputs to a neuron. Next, find $V(t)$ when this current $I(t)$ is applied to the circuit of Fig. 2.

Problem 5

Consider designing a coin dispenser for an amusement park. The unit of currency at this amusement park is *ryo*. A j ryo coin is referred to as a j-coin. The coin dispenser should dispense coins whose total amount exactly matches with each payment request.

The simplest way, for a payment request, is to repeatedly dispense a coin of the largest possible value. This will be called the greedy coin-dispensing algorithm.

(1) Assume that there are three kinds of coins, 1-coin, 2-coin, and 5-coin, used in the amusement park and that the coin dispenser has infinitely many coins of each kind. Show that the greedy coin-dispensing algorithm always dispenses the minimum number of coins for an arbitrary payment request of N ryo.

(2) Assume that there are three kinds of coins, 1-coin, 4-coin, and 5-coin, in the coin dispenser. Show that the greedy coin-dispensing algorithm may fail to dispense the minimum number of coins.

(3) Assume that there are three kinds of coins, 1-coin, 4-coin, and 5-coin, used in the amusement park and that the coin dispenser has infinitely many coins of each kind. For an arbitrary payment request of N ryo, design a dynamic programming algorithm to compute the minimum number of coins to dispense.

(4) Let N be the total amount of coins preset in the coin dispenser. Let c_j be the number of j-coins in the coin dispenser and m be the largest j with positive c_j. Show that if the coin dispenser can always fulfill arbitrary K payment requests whose total amount does not exceed N ryo, then $m \leq \lceil N/K \rceil$ and

$$\sum_{j=1}^{i} jc_j \geq Ki \qquad (i = 1, 2, \ldots, m-1) \qquad\qquad (*)$$

must hold, where $\lceil N/K \rceil$ is the smallest integer that is at least N/K.

(5) Under the above condition $(*)$, suppose that the coin dispenser employs the greedy coin-dispensing algorithm to dispense coins for a payment request with any amount up to N ryo. After this dispensation, let d_j be the number of j-coins left in the coin dispenser and l be the largest j with positive d_j. Show that

$$\sum_{j=1}^{i} jd_j \geq (K-1)i \qquad (i = 1, 2, \ldots, l-1)$$

holds. Furthermore, from this result, show that the coin dispenser, under the condition $(*)$, can always fulfill K payment requests whose total amount does not exceed N.