This booklet is an informal English translation of the original examination booklet.

- Answer three problems out of Problem 1 ~ Problem 5.
- Answer in Japanese or English.
Problem 1

A vector $p = (p_1, p_2, \ldots, p_n)$ is called a stochastic vector if it satisfies $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$ ($i = 1, 2, \ldots, n$). The entropy of a stochastic vector $p$ is defined as

$$H(p) = -\sum_{i=1}^{n} p_i \log p_i,$$

where we use the convention that $0 \log 0 = 0$. We denote by $\sigma_k(p)$ the sum of the $k$ largest elements of $p$.

A square matrix $A = (a_{ij} | i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ is called a doubly stochastic matrix if it satisfies

$$a_{ij} \geq 0 \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n),$$
$$\sum_{i=1}^{n} a_{ij} = 1 \quad (j = 1, 2, \ldots, n),$$
$$\sum_{j=1}^{n} a_{ij} = 1 \quad (i = 1, 2, \ldots, n).$$

Answer the following questions.

(1) Suppose that stochastic vectors $p, q$ and a doubly stochastic matrix $A$ satisfy $q = pA$. Prove that $H(q) \geq H(p)$.

(2) Suppose that stochastic vectors $p, q$ and a doubly stochastic matrix $A$ satisfy $q = pA$. Prove that $\sigma_k(q) \leq \sigma_k(p)$ for $k = 1, 2, \ldots, n$.

(3) Suppose that stochastic vectors $p, q$ satisfy $\sigma_k(q) \leq \sigma_k(p)$ for $k = 1, 2, \ldots, n$.

Prove that there exists a doubly stochastic matrix $A$ that satisfies $q = pA$. 
Problem 2

In the $n$-dimensional real space $\mathbb{R}^n$, a vector $c \in \mathbb{R}^n$ and a positive definite symmetric matrix $A$ determine an ellipsoid $\{ x \mid (x - c)^\top A (x - c) \leq 1 \}$, where $c$ is called its center. Here, $^\top$ means the transpose of a vector. We denote the components of a vector $x$ by $x_1, x_2, \ldots, x_n$.

Let $E$ be an ellipsoid whose center is at the origin. Let $K$ be one half of the ellipsoid $E$ obtained by cutting $E$ with a hyperplane $H$ through the origin. Let $F$ be the ellipsoid of minimum volume containing $K$. We denote the ratio of the volumes of $F$ and $E$ by $\rho = \frac{\text{vol}(F)}{\text{vol}(E)}$.

Answer the following questions.

(1) Obtain the center of $F$ and the value of $\rho$ when $E, H, K$ are given by

$$E = \{ x \mid \sum_{i=1}^{n} x_i^2 \leq 1 \},$$
$$H = \{ x \mid x_n = 0 \},$$
$$K = \{ x \mid \sum_{i=1}^{n} x_i^2 \leq 1, x_n \geq 0 \}.$$  

(2) For the value of $\rho$ obtained in (1), show that $\log \rho$ converges to 0 as $n \to \infty$, and discuss the rate of the convergence.

(3) Explain why $\rho$ in the general case is a function of $n$ independent of $E$ and $H$. 
Problem 3

(1) Equally-spaced straight parallel lines are drawn on a plane as in the figure. The distance between two adjacent lines is 1. A needle of length 1 is thrown at random on the plane. Obtain the probability $p$ that the needle intersects one of the lines.

(2) Consider an experiment where the needle is thrown $n$ times and $X$, the number of times the needle intersects a line, is recorded. Obtain the variance of $X/n$. What distribution does the distribution of $\sqrt{n} \left( \frac{X}{n} - p \right)$ converge to as $n \to \infty$?

(3) A regular $m$-gon ($m \geq 3$) of side length 1 is thrown at random on the plane. Let $Y$ be the number of intersections of the boundary of the polygon with the lines. Obtain the expectation of $Y$. 
Problem 4

For two square matrices $P$ and $Q$ of the same size, let $[P, Q]$ denote the matrix $PQ - QP$.

Given two real symmetric matrices $A$ and $B$ of the same size, consider the initial value problem of the differential equation on a real matrix $U(t)$:

$$\frac{d}{dt} U(t) = [A, U(t)BU(t)^\top] U(t),$$

$$U(0) = I.$$

Here, $I$ stands for the identity matrix and $\top$ denotes the transpose of a matrix.

Answer the following questions.

1. Show that $U(t)^\top U(t) = I$ for all $t > 0$.

2. Show that the matrix $X(t) = U(t)BU(t)^\top$ satisfies

$$\frac{d}{dt} \text{Tr}(AX(t)) = \text{Tr}([A, X(t)] [A, X(t)]^\top)$$

for all $t > 0$. Here, Tr denotes the trace (the sum of the diagonal elements) of a matrix.

3. Show that

$$\frac{d}{dt} \text{Tr}(AX(t)) \geq 0$$

for all $t > 0$ and that

$$\frac{d}{dt} \text{Tr}(AX(t)) \to 0$$

as $t \to \infty$.

4. Suppose that $A$ is a diagonal matrix whose diagonal elements are distinct real numbers. Show that $X(t)$ converges, as $t \to \infty$, to a diagonal matrix whose diagonal elements are the eigenvalues of $B$. 
Problem 5

Let \( P_0, P_1, \ldots, P_{N-1} \) be \( N \) (\( \geq 4 \)) points located clockwise on a circle, of which the center is at the origin \((0,0)\) of an orthonormal coordinate system. Suppose that the coordinates of each point \( P_i \) are given by a pair of rational numbers \((x_i, y_i)\). You may assume that the elementary arithmetic operations (addition, subtraction, multiplication, division and comparison) over rational numbers can be executed in unit time. Design an algorithm of time complexity linear in \( N \) for each of the following problems.

(1) Determine whether there exist, among the points \( P_0, P_1, \ldots, P_{N-1} \), two points at opposite ends of a diameter of the circle.

(2) Determine whether there exist, among the points \( P_0, P_1, \ldots, P_{N-1} \), four points that form a rectangle.

(3) Find four points out of \( P_0, P_1, \ldots, P_{N-1} \) that form a rectangle of maximum area.