数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成１４年８月２７日（火）　１３：００〜１６：００

５問出題，３問解答

○ This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 ∼ Problem 5.**
- **Answer in Japanese or English.**

Problem 1

Let f be a real-valued function defined by

$$f(X) = \log(\det X)$$

for a positive-definite symmetric matrix X, where “det” denotes the determinant of a matrix and “log” the natural logarithm. We consider an inequality

$$f(A) + f(B) \leq 2f\left(\frac{A+B}{2}\right), \qquad (*)$$

where A and B are positive-definite symmetric matrices of order n.

Answer the following (1), (2), and (3).

(1) Prove $(*)$ when $n = 1$.

(2) Prove $(*)$ when B is the identity matrix.

(3) Prove $(*)$ for general A and B.

Problem 2

In a hotel, n guests deposited their hats to a clerk. The clerk lost the records and decided to return these n hats in a random order. Under these circumstances, let X_n denote the number of guests with their hats correctly returned and let $p_n(k)$ denote the probability of the event $X_n = k$.

Answer the following (1), (2), (3), and (4).

(1) Compute the expected value and the variance of X_n.

(2) Show that $p_n(0)$ is written as

$$p_n(0) = 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!}\right).$$

(3) For each $k = 1, \ldots, n$ show that $p_n(k)$ is written as

$$p_n(k) = \frac{1}{k!} p_{n-k}(0),$$

where $p_0(0) = 1$.

(4) For each $k = 0, 1, 2, \ldots$ show the following convergence:

$$p_n(k) \to \frac{1}{k!}\mathrm{e}^{-1} \qquad (n \to \infty).$$

Problem 3

Answer the following (1), (2), and (3) on the linear programming problem

$$
\begin{array}{llll}
(\mathrm{P}_k) & \text{minimize} & x_1 + x_2 + \cdots + x_k & \\
& \text{subject to} & kx_1 & \geq 1, \\
& & x_1 + kx_2 & \geq 1, \\
& & x_1 + x_2 + kx_3 & \geq 1, \\
& & \vdots \quad\ \vdots \qquad\quad \ddots & \vdots \\
& & x_1 + x_2 + \cdots + x_{k-1} + kx_k & \geq 1
\end{array}
$$

with k real variables $x_1, x_2, \ldots, x_k$.

(1) Show the dual problem to the linear programming problem (P_k).

(2) Find the optimal value α_k of the linear programming problem (P_k).

(3) Show that α_k is monotone decreasing with respect to k, and find $\lim_{k\to\infty} \alpha_k$.

Problem 4

Suppose that an (x, y, z) orthonormal coordinate system is fixed to the three-dimensional space. Let C be the cylindrical surface having the x-axis as the axis and the radius a and lying in the range of $-a < x < a$, that is,

$$C = \{(x, y, z) \mid y^2 + z^2 = a^2,\ -a < x < a\}.$$

Let $f(x)$ be a differentiable function which is defined on $-a < x < a$ and which satisfies $0 < f(x) \leq a$. Furthermore, let S be the surface obtained by revolving the curve

$$y = f(x), \quad z = 0$$

around the x-axis, that is,

$$S = \{(x, y, z) \mid y^2 + z^2 = f(x)^2,\ -a < x < a\}.$$

For any point p on C, let $h(p)$ denote the point of intersection between S and the line segment dropped from p perpendicularly to the x-axis. Now, we generated points on C with the uniform density d, and mapped these points by the mapping h onto S. Then the resulting points were distributed on S with the same uniform density d. Answer the following (1) and (2).

(1) Find the differential equation that should be satisfied by $f(x)$.

(2) Find a general shape of the surface S.

Problem 5

Along a circular track, there are N gas stations which are numbered clockwise from 1 up to N. At station i, there are p_i gallons of petrol available. To race from station i to its clockwise neighbor one needs q_i gallons of petrol. Assume

$$\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i$$

and consider to race a complete lap by a car with an empty fuel tank. Let S_{ij} be the set of stations lying clockwise along the track from i to j (including i and not including j), and define

$$\mathrm{gain}(i,j) = \sum_{k \in S_{ij}} (p_k - q_k).$$

Answer the following (1) and (2).

(1) Use "gain" to describe a necessary and sufficient condition for a car to race the complete lap starting from station 1.

(2) Show that there always exists a station from which a car can race a complete lap.