数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成１３年８月２８日（火）　１３：００～１６：００

５問出題，３問解答

○ This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 $\sim$ Problem 5.**
- **Answer in Japanese or English.**

Problem 1

Let X, Y be independent uniform random variables over the interval $[0,1]$. For $0 < c < 1$, let a random variable Z_c be defined by

$$Z_c = \begin{cases} X, & \text{if } X \geq c, \\ Y, & \text{otherwise.} \end{cases}$$

Note that Z_c corresponds to the following procedure: If X is greater than or equal to c, we are satisfied and stop; otherwise we discard X and further observe Y. Similarly for a subset A of the interval $[0,1]$, let a random variable Z_A be defined by

$$Z_A = \begin{cases} X, & \text{if } X \in A, \\ Y, & \text{otherwise.} \end{cases}$$

Answer the following (1), (2), and (3).

(1) Obtain the probability density function of Z_c and the expected value $\mathrm{E}[Z_c]$ of Z_c.

(2) Obtain the maximizing value $c = c^*$ of $\mathrm{E}[Z_c]$ and the maximized expected value $\mathrm{E}[Z_{c^*}]$.

(3) Show that $\mathrm{E}[Z_{c^*}] \geq \mathrm{E}[Z_A]$ for every A.

Problem 2

Let S_n be the convex cone defined by

$$S_n = \left\{ \boldsymbol{x} = (x_0, x_1, \ldots, x_n)^\top \in \mathrm{R}^{n+1} \,\middle|\, x_0 \geq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \right\}.$$

Answer the following (1), (2), (3), (4), and (5).

(1) Describe the shape of the convex cone S_2.

(2) Show that for any vector $\boldsymbol{x} \in \mathrm{R}^{n+1}$, the property

$$\boldsymbol{x} \notin S_n \Leftrightarrow \exists \boldsymbol{y} \in S_n, \boldsymbol{x}^\top \boldsymbol{y} < 0$$

holds.

(3) Let A be a matrix in $\mathrm{R}^{m \times n}$ and $\boldsymbol{b}$ a vector in R^m. Describe the Karush-Kuhn-Tucker condition (Kuhn-Tucker condition) of the convex quadratic programming problem

$$\text{min. } z_1^2 + \cdots + z_n^2 \quad \text{s.t. } A\boldsymbol{z} \geq \boldsymbol{b}$$

where $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^\top$.

(4) Let X_r be the region $\{\boldsymbol{x} = (x_0, x_1, \ldots, x_n)^\top \in \mathrm{R}^{n+1} \mid A\overline{\boldsymbol{x}} \geq \boldsymbol{b},\ \overline{\boldsymbol{x}} = (x_1, \ldots, x_n)^\top,$ $x_0 = r\}$. Assume that there exists a vector $\boldsymbol{y}$ satisfying the condition $\boldsymbol{y} \geq \boldsymbol{0}$ and $\boldsymbol{y}^\top \boldsymbol{b} > r||A^\top \boldsymbol{y}||$. Using the fact that the vector $(||A^\top \boldsymbol{y}||, -\boldsymbol{y}^\top A)^\top$ belongs to the convex cone S_n, show that $S_n \cap X_r = \emptyset$. Here we define that $||\boldsymbol{u}|| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$ for any vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^\top$.

(5) Suppose that the region X_r defined in (4) satisfies $r > 0$ and $X_r \neq \emptyset$. Show the property

$$S_n \cap X_r = \emptyset \quad \Rightarrow \quad \exists \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{y}^\top \boldsymbol{b} > r||A^\top \boldsymbol{y}||$$

holds.

Problem 3

Prove the following identity

$$\lim_{n\to\infty}\int_0^1\cdots\int_0^1 \frac{n}{x_1+x_2+\cdots+x_n}\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n = 2.$$

Problem 4

We want to construct a slide having width D around a cylinder with radius R and with angle α with respect to the horizontal plane. As shown in Fig. 1, let ℓ' be the line drawn on the unfolded surface of the cylinder such that the angle between ℓ' and the horizontal line is equal to α, and let ℓ be the corresponding curve on the cylinder. As shown in Fig. 2, let P be a point moving along ℓ, and let $\overline{\mathrm{PS}}$ be the line segment with length D perpendicular to the surface of the cylinder. As P moves along ℓ, the line segment $\overline{\mathrm{PS}}$ sweeps a surface. This surface forms the floor of the slide. We construct this slide from an iron plate. For this purpose, answer the following (1), (2), (3), (4), and (5).

(1) As shown in Fig. 2, we fix the (x, y, z) orthogonal coordinate system in such a way that the z axis coincides with the axis of the cylinder, and the curve passes through the point $(R, 0, 0)$. Let s be the arc length from $(R, 0, 0)$ to P along ℓ. Express the (x, y, z) coordinates of P using α and s.

(2) Express the radius R' of the curvature of the curve ℓ using α and s.

(3) Let Q be the point on the line segment $\overline{\mathrm{PS}}$ such that the distance from P to Q is u. Find the length t of the curve traced by Q when P moves a distance s along ℓ.

(4) To make the floor of the slide, we cut an iron plate along circular arcs having a common center O′ with radius R' and $R' + D$, as shown in Fig. 3. Let P be a point on the smaller circular arc, and let Q′ be the point on the half line emanating from O′ and passing through P such that Q′ is at distance u from P in the direction opposite to O′. Find the length t' of the arc traced by Q′ when P moves a distance s along the smaller circular arc.

(5) In order to construct the floor of the slide, we fix the smaller circular arc of the iron plate along ℓ, and hit the plate to expand. Suppose that we pull the plate in the direction parallel to the circular arcs while hitting, and hence the plate expands only in the direction parallel to the circular arcs. Find the ratio of the thickness of the plate at Q to the thickness at P in the final state of the floor of the slide.

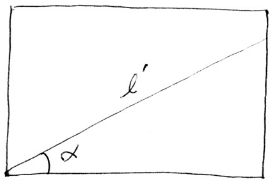

Fig. 1

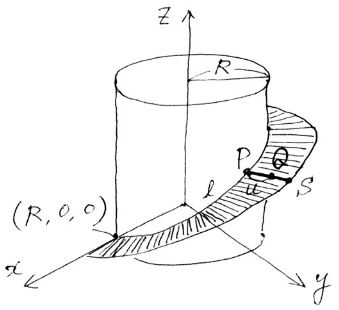

Fig. 2

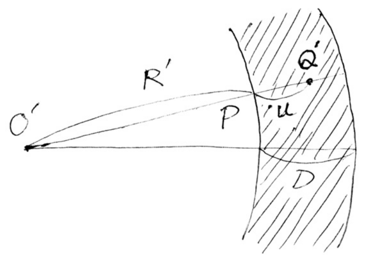

Fig. 3

Problem 5

Consider to permute an integer sequence using a stack (starting from an empty one). There are two operations, S and X, where S stands for "move an integer from the input integer sequence into the stack", and T for "move an integer from the stack into the output". A list of S and X is called operation sequence. For example, applying the operations in the operation sequence SSXSSXXX from left to right can permute 1234 to 2431.

Answer the following (1), (2), and (3).

(1) Enumerate all possible permutations which can be obtained from 1234 using a stack.

(2) Show that it is possible to obtain a permutation $p_1 p_2 \ldots p_n$ from $12\ldots n$ using a stack if and only if there are no indices $i < j < k$ such that $p_j < p_k < p_i$.

(3) Design your algorithms

  (i) to compute the output integer sequence by applying a given operation sequence $\mathrm{O}_1\mathrm{O}_2\ldots\mathrm{O}_m$ ($\mathrm{O}_i \in \{\mathrm{S}, \mathrm{X}\}$) to $12\ldots n$ ($n \geq m$), and

  (ii) to generate an operation sequence which can permute $12\ldots n$ into a given integer sequence $p_1 p_2 \ldots p_n$.

  Code your algorithms in a programming language and analyze their time complexity.