Written Exam

10:00 – 12:30, February 7, 2017

Entrance Examination (AY 2017)

Department of Computer Science
Graduate School of Information Science and Technology
The University of Tokyo

Notice:

(1) Do not open this problem booklet until the start of the examination is announced.

(2) Answer the following 4 problems. Use the designated answer sheet for each problem.

(3) Do not take the problem booklet or any answer sheet out of the examination room.

Write your examinee's number in the box below.

Examinee's number	No.

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Problem 1

For a positive integer p, the p-norm $\|\boldsymbol{x}\|_p$ of an n-dimensional real vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is defined by

$$\|\boldsymbol{x}\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} .$$

Answer the following questions.

(1) Prove that

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2$$

holds for every n-dimensional real vector $\boldsymbol{x}$. You may use the Cauchy-Schwarz inequality:

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2$$

for any n-dimensional real vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Here $\boldsymbol{x} \cdot \boldsymbol{y}$ stands for the inner product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

(2) Define the p-norm $\|A\|_p$ of an $n \times n$ real matrix A by

$$\|A\|_p := \max_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p},$$

where $\boldsymbol{x}$ ranges over the set of n dimensional real vectors.

(2.1) Prove that if $\|A\|_p < 1$ then $\lim_{k \to \infty} \|A^k \boldsymbol{x}_0\|_p = 0$ for every n dimensional real vector $\boldsymbol{x}_0$.

(2.2) Suppose that A is an $n \times n$ real symmetric matrix. Prove that $\|A\|_2$ is the maximum of the absolute values of the eigenvalues of A.

(3) Consider solving an n-dimensional linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A is a non-singular real symmetric matrix, and $\boldsymbol{x}$ and $\boldsymbol{b}$ are unknown and constant real vectors, respectively.

Given an initial vector $\boldsymbol{x}^{(0)}$, the vector $\boldsymbol{x}^{(j)}$ $(j = 1, 2, \ldots)$ is computed by

$$\boldsymbol{x}^{(j)} = \boldsymbol{b} + (I - A)\boldsymbol{x}^{(j-1)},$$

where I stands for the identity matrix.

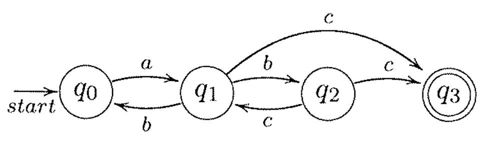
Give a necessary and sufficient condition on A such that the sequence $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots$ converges to the true solution for every initial vector $\boldsymbol{x}^{(0)}$.

Problem 2

For a non-deterministic finite automaton M over an alphabet Σ, we write $\mathcal{L}(M) \subseteq \Sigma^*$ for the set of words accepted by M. We write $|w|$ for the length of the word w, and write $\mathbb{N}$ for the set of non-negative integers.

Answer the following questions.

(1) Consider the non-deterministic finite automaton M_0 depicted below, where q_0 is the start state, and q_3 is the only final state. Give $x, y, z \in \{a, b, c\}^*$ that satisfy all of the following conditions: (i) $xyz = abcc$, (ii) $|y| > 0$, and (iii) $xy^n z \in \mathcal{L}(M)$ for every $n \in \mathbb{N}$.



(2) Prove that, for every non-deterministic finite automaton M consisting of k states and for every $w \in \mathcal{L}(M)$ such that $|w| \geq k$, there exist x, y, and z that satisfy all of the following conditions: (i) $xyz = w$, (ii) $|y| > 0$, (iii) $|xy| \leq k$, and (iv) $xy^n z \in \mathcal{L}(M)$ for every $n \in \mathbb{N}$.

(3) Prove that there exists no non-deterministic finite automaton M such that $\mathcal{L}(M) = \{a^m b^n \mid m, n \in \mathbb{N},\ 0 < m < n\}$. You may use the fact proved in question (2).

Problem 3

Consider the problem of sorting an array of integers using the heapsort algorithm. Assume that heapsort brings the minimum element to the front.

Answer the following questions.

(1) Heapsort consists of two phases. Explain what is to be performed in each phase.

(2) Consider sorting of the following array using heapsort.

$$(3, 8, 1, 5, 4, 9, 7)$$

Draw the tree structure of the heap just after the first phase.

(3) Answer the time complexity of each phase when sorting an array of length n. Explain the reason.

(4) Answer the time complexity of obtaining a sorted list of the smallest k elements from an array of length n using heapsort. Explain the reason.

(5) The execution time of heapsort on modern computer systems is often longer than that of some other sort algorithms such as quicksort and mergesort when sorting a large array. Explain a possible reason.

Problem 4

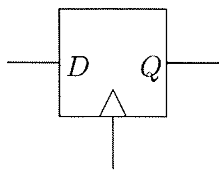
Let us consider implementing a linear recurrence sequence generator as a sequential circuit. Assume that the clock is an ideal rectangular wave without skew, and that the gate delays are negligible. The clock signal must be used for the *Clock* inputs of all the flip-flops, and must not be used otherwise.

Let m and $0 < j_1 < j_2 < \cdots < j_m \leq 256$ be positive integers. Given initial values x_0, x_1, $\ldots$, x_{255}, the value of x_n for $n \geq 256$ is defined by the recurrence equation:

$$x_n = x_{n-j_1} \oplus x_{n-j_2} \oplus \cdots \oplus x_{n-j_m},$$

where $x_i \in \{0, 1\}$, and $\oplus$ represents the exclusive-or (XOR) operator.

You can use AND, OR and NOT gates and positive-edge-triggered D flip-flops in your circuit design. Recall that a positive-edge-triggered D flip-flop has two inputs D and *Clock*, and one output Q. At the positive-edge of the *Clock* signal, the D input is sampled and stored as the current value of Q. The value of Q is kept unchanged until the next positive-edge of the *Clock* signal. A flip-flop is depicted as the following figure in this problem.

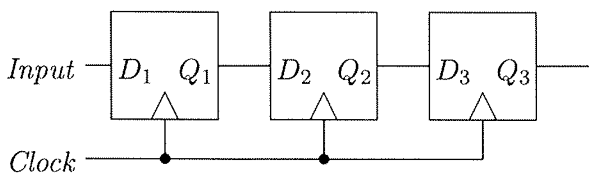


Answer the following questions.

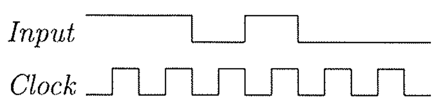
(1) Design a 2-input XOR circuit with AND, OR and NOT gates.

(2) Consider the following circuit and the *Input* and *Clock* signals depicted by the timing diagram below. Assume that the initial values of Q_1, Q_2 and Q_3 are zeros. Depict the timing diagram for Q_1, Q_2, and Q_3 along with the *Input* and *Clock* signals.

Circuit:



Timing diagram:



(3) Suppose that we have 512 positive-edge-triggered D flip-flops X_1, X_2, $\ldots$, X_{256} and C_1, C_2, $\ldots$, C_{256}. Assume that initially X_i stores the value of x_{256-i} and C_i stores 1 if $i \in \{j_1, j_2, \ldots, j_m\}$ and 0 otherwise, for $i = 1, 2, \ldots, 256$. Design a circuit so that X_i stores the value of $x_{256-i+k}$ at the kth clock cycle.

(4) Modify the circuit you have answered in question (3) as follows, in order to enable initialization of the flip-flops. It is enough to show only the difference. Add three inputs W, X_0 and C_0. At the positive edge of *Clock*, the circuit should execute the following. If input signal W is 0, the value of X_{i-1} is moved to X_i, and the value of C_{i-1} is moved to C_i, for $i = 1, 2, \ldots, 256$. If input signal W is 1, the circuit works as described in question (3).

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